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REINFORCED CONCRETE DESIGN

REINFORCED CONCRETE DESIGN

BY

G. P. MANNING, M.ENG., ASSOC.M.INST.C.E.

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PREFACE TO SECOND EDITION

The twelve years that have elapsed since the appearance of the first edition have merely confirmed the author's expressed opinions. Although much of the book has been re-written, the revisions are mostly arithmetical, due to variations of the allowable working stresses and variations in the modular ratio. The general arrangement and sequence of chapters remain unaltered.

Parts of Chapter XXIV first appeared in *Engineering* and are reproduced by courtesy of the Editor of that journal.

Many of the new diagrams are simplified working details from structures designed and built by Peter Lind and Co., Ltd., and are included by kind permission of that firm.

Many of the new diagrams have been prepared by Mr. Edward Mills, A.F.A.S.

The author is most deeply indebted to his late wife Kathleen Favell Manning for her continued encouragement and support.

G.P.M.

PREFACE

This work aims at establishing methods of design suitable for use in a Civil Engineer's Office, and is based on the methods which the author has found in use and which he has himself used in actual designs.

The First Part treats of the cross-sections of members and the stresses produced. Each problem is first viewed from the theoretical standpoint in accordance with the usual assumptions of elastic deformation; allowances are then made to adjust this academic treatment to the practical material and finally the results are reduced to simple numerical factors for practical application, illustrated by examples. In working out the theoretical analysis the shortest route has been chosen, generally by means of algebra and a little calculus, as graphical methods are longer and sometimes very obscure.

The Second Part deals with the question of members as a whole, on lines somewhat similar to those employed in the First Part.

The Third Part is a discussion of complete structures. Owing to lack of space, only a few examples are given in this part.

There is no need to emphasize the fact that economy of cost coupled with a reasonable factor of safety is an essential of good design. Brevity and speed are equally essential to practical design. Before any method, process or formula can be adopted in practice it must be so shortened and simplified as to be capable of being used and checked in the minimum time. Apart from the fact that designs and tenders must all be completed in a certain stipulated time, long and laborious mathematical processes increase office costs, give greater opportunities for error, and discourage and fatigue the designer. For every member constructed, some twenty or thirty designs are prepared (owing to non-acceptance of competitive tenders, changes in layout, preparation of alternative schemes, etc.). Speed is therefore the very first essential of practical design and all long-winded methods or formulae, particularly those introducing " blind " functions, must be avoided. In reinforced concrete design, speed, simplicity, accuracy and economy are synonymous terms.

The theory of reinforced concrete structures, as such, is non-existent. The best that modern theory can offer as a substitute is the purely imaginary conception of an ideal elastic structure, and it must be admitted that there are radical differences between a so-called " monolithic " reinforced concrete building and a truly continuous elastic structure.

Most of our present knowledge of practical design was developed by specialist firms in competitive work, amongst whom mention must be made of the well-known Hennebique firms. As competition grew keener, designers used smaller and smaller sections, until failures occurred. We can therefore look upon competitive design as a gigantic series of trial-and-error experiments on the largest scale, and present day methods of design are the collective results of these experiments. Of late years, a new field of information has been opened by the Americans in the shape of experiments on actual reinforced concrete structures (not models). The inevitable drawback to these is the impossibility of loading to destruction, as the distribution of stress varies with the intensity of loading, making it impossible to forecast the ultimate strength of the structure from readings taken under working load or under test load. The results of the American investigations into flat-slab floors are worthy of particular mention. Not only do the observed bending moments differ by fifty per cent. or more, from the calculated values, but the amount and distribution of the observed moments can be varied within wide limits by adopting different arrangements of the reinforcing bars. Both these phenomena, the divergence between the calculated stresses in an elastic structure and the actual stresses in a reinforced concrete structure, and the possibility of re-arranging the stresses by re-arranging the steel, have been known and used by experienced designers for years, and, being amply supported by standard practice, have led the author to formulate his Principle of Greatest Strength. It must be clearly understood that the theory of the elastic structure is quite unable to explain either of these phenomena, and what is more surprising is that most writers on the subject appear to be totally unaware of these fundamental and very vital principles.

The only logical theory to apply to reinforced concrete design is that which is based on the conditions which exist in an actual structure *when it has just reached its ultimate load*, and there are very strong indications that such a theory will be evolved during the next twenty or thirty years. As, however, this book is intended for use to-day, the author has given the usual theory of elastic structures while endeavouring everywhere to point out that this theory does not apply exactly and can only be relied on as furnishing a very rough indication of the stresses in actual structures as built.

The design of a structure, in the full and modern sense of the word, includes the thinking out of the supply of materials, and methods of construction. Some designers are given to drawing very light and complicated structures which involve the minimum outlay of materials, but which necessitate an incredible expenditure of time, labour and supervision during erection. The author of this work is a staunch advocate of simple and straightforward design, and has never seen or heard of any job which was erected quickly, being erected at a loss. It is the complicated jobs where one part holds up the construction of another, and which hang fire and drag along wearily from month to month, which prove unsatisfactory when completed, and which drive contractors into bankruptcy. The general principles of simple, quick and cheap design apply to all materials, timber, stone, steel, mass or reinforced concrete, and the author, owing to lack of space, has had to omit reference to them here and confine his remarks to those problems of design which are peculiar to reinforced concrete. On no account must a design be altered once erection has commenced. There is no surer way of wasting time and money than holding up a job for minor alterations or " improvements " in design.

Our knowledge of reinforced concrete structures has been very seriously hindered and obscured by the development of the theory of the elastic structure. The absolute mathematical certainty of the figured results when the material is assumed to be elastic (in the full scientific meaning of the term), appears to have fascinated many academical writers on the subject, and has completely blinded them to the fact that the resemblance between a concrete structure and a perfect elastic structure is very slight.

The author's best thanks are due to Miss K. F. Everett, to Miss E. M. Shepley for assistance with the larger diagrams, and most particularly to Mr. P. C. G. Hausser, B.Sc., A.C.G.I., for assistance in reading and checking the manuscript and proofs.

Appendix A, which contains the Flat-Slab Regulations of the City of Chicago, is reproduced from the *Concrete Engineers' Handbook* by Hool and Johnson, by kind permission of the Hill Publishing Company. Other acknowledgments will be found in the text.

G. P. M.

102 Fordhook Avenue
Ealing Common
London, W.5.

March 1924.

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NOTATION

Owing to the large number of different problems attacked, it is quite impossible to confine one particular symbol or letter of the alphabet to each particular case, and the same symbols have to be used to represent different values in different problems. Attempts have been made to introduce various Standard Notations but the author has never seen the necessity for so doing.

In every problem attacked in the text the values of all symbols employed have been stated in each separate case. Each Table of Reference shows by notes and diagrams the meanings of the letters and symbols employed on it. Moreover, in the text, wherever possible, the terms " lever-arm", " effective depth" , " width of flange ", etc., have been written in full instead of employing the symbols " *jd* ", " *d* ", " *B* ", etc.

Unless stated to the contrary, all working examples are expressed in pounds and inches.

The list which follows, shows only those symbols in more general use in the text, which have been based on the system in use in American text-books.

A_s = Area of Tensile Steel in Beams.
A_s' = Area of Compression Steel in Beams.
a = (In Column Design) Overall Depth of Section
a = (In Arch Design) Half the Span.
B = Width of Flange in T-Beam Sections.
b = Breadth of Rectangular Beam Sections.
b = (In Arch Design) The Rise.
b' = Width of Rib in T-sections.
D = Diameter.
d = Effective Depth.
d' = Distance of Centre of Compression Steel from Compressed Edge.
E_c = Elastic Modulus of Concrete.
E_s = Elastic Modulus of Steel.
EI = Deflection Coefficient.
ϵ or e $\Big\}$ Eccentricity of Loading Measured from the Centre Line of the total Concrete Section.
F = Total Shear Force.
f_c = Maximum Fibre Stress in Concrete.
f_s = Stress in Tensile Steel.
f_s' = Stress in Compression Steel.
h = Horizontal Spacing of Stirrups.
= Moment of Inertia.

i = Effective Area of one Stirrup.
i_a = Inclination of Tangent at a, etc.
j = A Fraction.
jd = Lever Arm.
K_a K_b etc. = Stiffness Factors (Chapter XIX).
k = A Fraction.
ka = Distance to Centre of Steel from Bottom Outside Edge.
k' = A Fraction.
$k'a$ = Distance to Centre of Steel from Top Outside Edge.
L = Total Load.
l = Span.
M = Bending Moment.
M_a = Bending Moment at a, etc.
m = Ratio of Elastic Moduli $= \frac{E_s}{E_c}$.
n = Fraction Showing Depth of Neutral Axis.
na = Depth of Neutral Axis in Columns.
nd = Depth of Neutral Axis in Beams.
P = Total Compressive Thrust.
p = (In Parts I and II) Percentage Area of Steel in Column Sections.
p = (In Part III) Intensity of Pressure.
p_s = Percentage Area of Tensile Steel (on Area bd)
p_s' = Percentage Area of Compression Steel.
R = A Factor Showing the Moment of Resistance of a Rectangular Beam Section $= \frac{M}{bd^2}$
(Also sometimes is used to = Radius).
r = Radius.
s_c = Shear Compressive Stress in Concrete.
s_p = Punching Shear Stress.
s_s = Shear Tensile Stress on Steel.
T = Total Tensile Pull.
t = Thickness of Concrete Slab or Flange.
t_c = Thickness of Concrete Shell.
t_s = Thickness of Steel Shell.
v = A Factor see Fig. 157.
W = Point Load.
w = Distributed Load.
$\left.\begin{matrix}x\\y\end{matrix}\right\}$ = Cartesian Co-ordinates.
α, β, θ, ψ etc. Angles in Radians.
Φ = Coefficient of Expansion.
Δ = Deflection.
ρ = Radius of Curvature.

CHAPTER I

BASIS OF DESIGN

Structures.—In general a structure consists of a collection of members which carry external loads at specified points and transmit them to certain other points known as the foundation.

Members.—As a rule members are thin pieces of material whose thickness for purposes of calculation is small compared with their length, ranging from one-tenth to one-thirtieth.

For general usefulness and economy they are usually straight and mostly, although not always, of constant gross cross-section. Should the thickness of a member be large in proportion to its length, say one-half or one-third, the engineer is reduced to following the lines of some previous similar structure or making some sort of approximate guess as present-day analysis cannot solve such cases.

A member must be strong enough to resist

(a) external loads (superloads, wind loads, etc.),
(b) end thrusts, reactions and moments from other members.

The thrusts and moments due to (a) are fairly easy of calculation as the loads are specified beforehand.

In a structure of reinforced concrete, which is partly monolithic and to some extent elastic, the thrusts and moments due to (b) can be approximated to by comparing the actual structure in question with an ideal elastic structure of similar outline.

Design of Structures.—Knowing the loads required to be carried the engineer is therefore faced with two problems.

(1) to decide on the most economic type of structure.
(2) to design in detail each member of that structure.

The first of these two problems is usually complicated by many special local conditions peculiar to the particular structure in question and to solve it quickly and successfully comes only with experience.

The second problem is more straightforward and can be solved by analysis combined with a working knowledge of costs.

Although it is obviously necessary to solve the first before the second can be commenced, the two problems are by no means independent as some trial and error may be necessary before the best spacing of members is arrived at.

The author has thought it best to treat the second of these problems first.

Design of Members.—The strength at any point of a long thin member subject to direct load or bending moment, or a combination of these, can be determined by a consideration of the cross section at right-angles to the long axis of the member, i.e., the minimum cross section. The strength of a member subject to shearing forces depends not only on the minimum section but also on the longitudinal section in the vicinity.

The design of members therefore falls into two parts.

(1) to determine the exact thrusts, moments and shears at all points along the member.

(2) to supply adequate sections to resist them.

As in the design of structures, the first of these two must be worked out before the second can be commenced.

However, since the second can be treated as an independent problem by designing sections to carry a series of assumed thrusts, bending moments and shears, it is given first place in this work.

Strengths and Properties of Materials.—For a proper understanding of the subject the engineer should have a knowledge of the manufacture of steel and cement, of the sources of supply and properties of different aggregates and of the behaviour of concrete during mixing, placing, setting and under load. There are many works which treat of these subjects* and reference to them here will be avoided as far as possible.

Although all these considerations, coupled with a philosophical determination of a factor of safety, are the fundamentals on which all design must be based, reference to them in the actual processes of practical design is impossible owing to the absolute necessity for speed and brevity. In this connection it should be remembered that in competitive design nine out of ten schemes are wasted because tenders based on them are not accepted. Owners and architects also are given to changing their minds as the work progresses, so that finally for every member actually constructed some ten to twenty preliminary designs may be made.

In design, therefore, it is necessary to assume simple and approximate properties as near to the actual values as possible.

Steel.—The reinforcement consists of mild steel bars (variations due to using other kinds of steel are discussed in Chapter XXXV). For purposes of design, steel is a perfectly elastic material in both tension and compression. Its elastic modulus is 30,000,000 lbs. per sq. in. As only small sections are employed the stiffness of the bars themselves is neglected.

Concrete.—Of all modern building materials concrete is the most variable. The elastic modulus under working stresses may vary from below 1,500,000 to above 5,000,000 lbs. per sq. in. in adjacent parts of the same structure. If the stresses are increased beyond the safe working stresses the elastic modulus falls continuously.

---

* e.g. *Concrete Plain and Reinforced*, Taylor and Thompson.

Specimens cut from parts of a completed structure may show crushing strengths ranging below 1000 lbs. per sq. in. to above 4000 lbs. per sq. in. It is essential that we assume some simple average values in practical design and it seems that an assumed value of 2,000,000 lbs. per sq. in. for the elastic modulus is somewhere approximately right. At the moment it is fashionable to vary the assumed value with the quality of the concrete and although there is no justification for such entirely fictitious " accuracy " the author has included curves for values ranging from 1,670,000 ($m = 18$) to 3,000,000 lbs. per sq. in. ($m = 10$). The tensile strength of concrete is about one-tenth of its compressive strength except where construction joints or shrinkage cracks occur where, of course, it is zero.

As far as design is concerned concrete is a perfectly elastic material under compressive stress, the safe stress and elastic modulus varying with the quality of the concrete. It is incapable of withstanding any tensile stress except when this is caused by shear stress. The safe beam shear stress is about one-tenth and the safe punching shear stress is about one-fifth of the safe compressive stress.

Reinforced Concrete.—It is assumed that plane cross-sections of members before stressing remain plane under load. The steel and concrete act in conjunction throughout, there being no relative movement of the bars in the concrete.

In those parts of the sections which are under direct tensile stress the concrete is assumed to have ceased to function and can be neglected as far as stress is concerned. Thus at all points where tension is developed a sufficient area of steel must be introduced adequately to carry tensile stresses set up (except diagonal shear tension).

Although the shrinkage stresses in many modern structures are higher than the structural stresses, all such stresses and stresses due to discontinuity of construction, inequalities in workmanship and materials, expansion and contraction under heat, etc., are neglected except in the case of encastré arches and chimneys. That is to say, for purposes of design, there is no initial stress in any member.

Working Stresses.—As indicative of modern outlook the following is typical of general building work :

CONCRETE (nominal 1 : 2 : 4 mix).

in compression	750 lbs. per sq. in.
in direct tension	nil.
in beam shear	75 lbs. per sq. in.
in punching shear	150 lbs. per sq. in.

MILD STEEL in tension 18,000 lbs. per sq. in.

BOND STRESS 100 lbs. per sq. in.

MODULAR RATIO 18.

For watertight, marine or very exposed work the stresses are usually kept down, partly on account of the certainty that the assumed water pressures will be fully realized and partly to keep down fine cracks. The following are fairly representative of stresses in such work.

CONCRETE (nominal 1 : 1½ : 3 mix).

in compression	750 lbs. per sq. in.
in direct tension	nil.
in beam shear	75 lbs. per sq. in.
in punching shear	150 lbs. per sq. in.

MILD STEEL in tension due to bending, 16,000 lbs. per sq. in.
in direct tension members, 13,500 lbs. per sq. in.

BOND STRESS 100 lbs. per sq. in.

MODULAR RATIO 15.

The compressive stress on the reinforcement is never in question as the surrounding concrete will fail before the steel fails (see Chapter XVI). It is usual to allow higher local stresses in columns when these are due to wind stresses. An increase of 20 to 25% in the working stresses under such conditions is fairly general.

For concretes made under exceptionally good working conditions, from exceptionally good materials or containing high percentages of cement it is usual to allow higher stresses according to the strengths indicated by test-cubes. A working stress on the concrete in compression of one-third of the crushing strength of the weakest test-cube is generally regarded as safe, the shear stresses being increased pro rata.

With richer mixes of concrete the value of the elastic modulus rises (and the shrinkage increases) and although the effective modular ratio is such a vague, complicated and variable value that any attempt at accuracy is impossible, the author gives a range of values in the chapters that follow.

Methods of calculation based on a parabolic stress-strain curve for concrete have been put forward, but not generally adopted.*

Cover.—Before we can estimate the effective depth of a section we must know what cover to allow for. It is clear from a study of the deterioration of old structures that pre-war ideas on cover are quite inadequate. The cover may be estimated from the values below.

In no case should the cover outside any bar be less than the diameter of the bar.

COVER (from outside of bar to concrete surface)

	I	II
In floor and wall slabs, not less than .	½″	¾″ to 1″
Main bars in beams, not less than . .	1″	1½″ to 2″
Main bars in columns, not less than .	1″	1½″ to 2″
Secondary bars in beams, not less than .	½″	1¼″ to 1½″
Links in columns, not less than . . .	½″	1¼″ to 1½″

---

* *Principles of Reinforced Concrete Construction* : Turneaure and Maurer.

The values in Column I are usual in competitive design, in work which is protected by plastering, by asphalte, by brickwork, etc., and are still used by many designers who cling to pre-war ideas on the subject (and who usually blame the contractor when the steel starts to show through the concrete). The values in Column II are for first-class permanent design, for watertight work or exposed structures; the higher values being called for in exposed marine work.

The proper clearances between neighbouring reinforcing bars are given in Chapter IV.

CHAPTER II

RECTANGULAR SECTIONS UNDER BENDING ONLY. NO COMPRESSION STEEL

Placing of Steel.—In a reinforced concrete structure since there is no relative movement of the reinforcement in the concrete the strain on the steel must be equal to that on the concrete immediately surrounding it. Now the elastic modulus of mild steel is 30,000,000 lbs. per sq. in. while that of concrete is about 2,000,000 lbs. per sq. in. Therefore the stress on the steel is about fifteen times the stress on the concrete in the immediate vicinity for compressive reinforcement. As the area of concrete in tension is to be neglected the steel must of itself supply all the tension.

It follows that the best results are obtained by placing the steel as near the outer edges of the section as possible since the greatest strains always occur there.

To be theoretically exact the edge of a bar nearer the centre of a beam has less stress than the edge farther away from the centre. However, as the area of steel employed is always small, it can be assumed that the stress on the cross-section of any bar is uniform and the total stress on the bar acts through the centre of the bar.

Symmetry.—In practice a beam carrying vertical loads is made symmetrical about a vertical centre line, so that vertical loads cause bending of the beam but produce no torsional effect.

Effective Depth.—All concrete area in tension is neglected, therefore, for purposes of design, the depth of a beam is measured from the compression edge to the centre of the tensile steel and not to the extreme edge of the concrete cover under the tensile bars. This depth is called the effective depth.

Analysis.

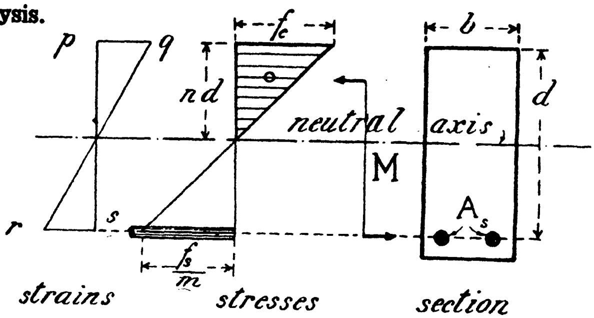


Fig. 1.

Let Fig. 1 represent a rectangular section subjected to a bending moment M lbs. ins.

$$\text{breadth} = b \text{ (inches)}$$
$$\text{effective depth} = d \text{ (inches)}$$

The section is reinforced with

A_s square inches of tensile steel

We have at once by definition

$$\text{percentage of tensile steel} = p_s = \frac{A_s}{b.d} \times 100.$$

Let the neutral axis occur nd from the compression edge. Then a straight line diagram $pqrs$ will represent to some scale the *strains* on the different fibres of material.

Let f_c represent the maximum fibre stress on the concrete.

We have $\dfrac{\text{Strain on Tensile Steel}}{\text{Strain on concrete at edge}} = \dfrac{rs}{pq} = \dfrac{d - nd}{nd} = \dfrac{1 - n}{n}$

by similar triangles.

As the ratio of elastic moduli $= \dfrac{E_s}{E_c} = m$

$$\frac{\text{Stress on Tensile steel}}{f_c} = \frac{f_s}{f_c} = \frac{m\,(1-n)}{n}$$

Total tension $= A_s \times f_s = A_s \,.\, m \dfrac{(1-n)}{n} f_c.$

Since the concrete stress varies uniformly from f_c to zero

Total compression $= \dfrac{1}{2} f_c\, bnd.$

As the section has to resist a bending moment only

Total tension = total compression

$$\text{i.e. } A_s m \frac{(1-n)}{n} f_c = \frac{1}{2} f_c\, bnd$$

putting in the value $\quad p_s = \dfrac{A_s}{b.d} \times 100$

we have $\quad \dfrac{p_s\, b\,.\,d}{100} \times \dfrac{m\,(1-n)}{n} f_c = \dfrac{1}{2} f_c\, bnd$

which gives $\quad p_s = \dfrac{50}{m} \cdot \dfrac{n^2}{(1 - n)} \quad \ldots\ldots \quad (1)$

Thus the depth of the neutral axis depends only on the percentage of reinforcement and the value of m, and is quite independent of the bending moment and of the stresses set up.

Practical values of n.—Assuming $m = 15$

By putting $\quad n = \cdot 1$

$= \cdot 2$

$= \cdot 3$ etc.

Values of p_s corresponding to these values of n are obtained and may be plotted as in Figure 2 (which gives also values for $m = 12$ and $m = 18$)

e.g. if $n = \cdot 5$

$$p_s = \frac{50}{15} \times \frac{\cdot 5^2}{(1 - \cdot 5)} = \frac{50}{30} = 1{\cdot}67\ \%.$$

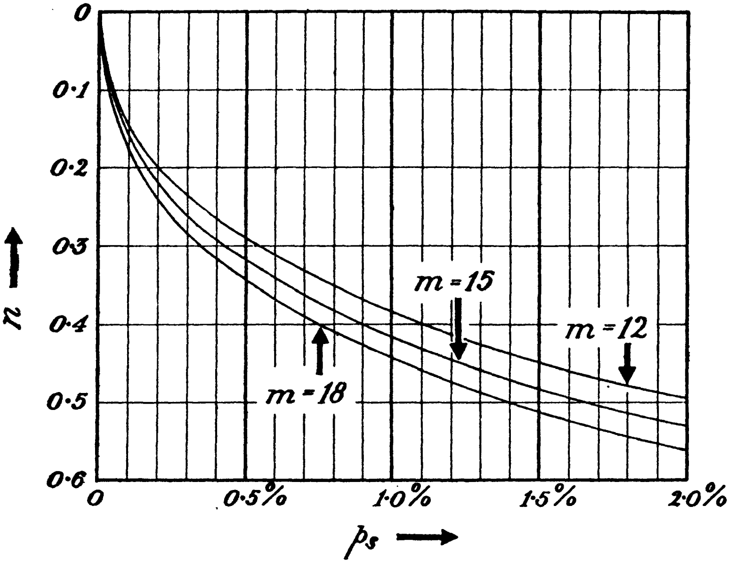


Fig. 2

Resistance Moments.—The centre of pressure of the stressed concrete area is distant $\frac{nd}{3}$ from the compression edge.

Total compression as before

$$= \frac{1}{2} f_c\, bnd$$

depth from centre of pressure of compression to tensile steel

$$= d - \frac{nd}{3}$$

Moment of Resistance = applied Bending Moment

$$\frac{1}{2} f_c\, bnd \times \left(d - \frac{nd}{3}\right) = M \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$$\text{also} \quad \frac{p_s \,.\, b \,.\, d}{100} \times f_s \times \left(d - \frac{nd}{3}\right) = M \quad . \quad . \quad . \qquad (2a)$$

If f_c, b, d and f_s are all expressed in units of pounds and inches then M will be in lbs.-ins.

Critical Values of n and p_s.—Before we can say that the section is not overstressed we must see that f_c does not exceed the safe working stress on the concrete and also that f_s does not exceed the safe working stress on the steel. If we supply a very small area of steel the value of f_s will reach and pass the safe stress before f_c reaches its full allowable value. And if we keep on increasing the moment applied to such a section the steel will fail by tension. If we supply a very large area of steel the section will fail by the concrete crushing out long before the steel is overstressed. For any given mix of concrete we can find the amount of steel required such that the section is equally strong in tension and in compression. Under these conditions the full working bending moment applied to the section will cause both f_c and f_s to reach their full allowable values at one and the same time. This condition gives us at once the value of n and from n we have p_s by equation (1). Suppose we have a section made of concrete which has a safe working stress of 890 lbs. per sq. in., and that m for this particular concrete is 17. Suppose the section is reinforced with steel which has a safe working stress of 19,000 lbs. per sq. in.

We have shown that $\frac{f_s}{f_c} = \frac{m(1 - n)}{n}$

If $f_s = 19{,}000$ and $f_c = 890$ with $m = 17$ then

$$\frac{19{,}000}{890} = \frac{17(1 - n)}{n}$$

This gives $n = 0{\cdot}443$

Also $p_s = \frac{50}{m} \frac{n^2}{(1 - n)} = \frac{50}{17} \frac{(0{\cdot}443)^2}{(1 - 0{\cdot}443)}$

$= 1{\cdot}035\%$ of steel

Therefore if a section made of such concrete is reinforced with 1·035% of steel it is equally strong in tension and compression.

If a section made of this particular concrete has less than 1·035% of steel it is weaker in tension than compression, and the maximum moment it will safely resist may be found from the expression in equation (2a).

$$M = \frac{p_s \,.\, b \,.\, d}{100} \times f_s\left(d - \frac{nd}{3}\right)$$

If a section made of this particular concrete has more than 1·035% of steel it is weaker in compression than tension and the maximum moment it will safely resist is found from equation (2).

$$M = \frac{1}{2} f_c\, b\, nd \times \left(d - \frac{nd}{3}\right)$$

The critical values of n and p_s that make the section equally strong in tension and compression are very important and the reader must be able to work them out at once for any given values of f_c, f_s and m.

If $f_c = 750$, $f_s = 18{,}000$ and $m = 18$
then $n = 0{\cdot}428$ and $p_s = 0{\cdot}89\%$.

Strength of the Section.—This is best illustrated by taking definite values for f_c, f_s and m.

If f_c must not exceed 750 lbs. per sq. in.
f_s „ „ „ 18,000 „ „ „ „
and $m = 18$

the critical values of n and p_s are as above

$$n = 0{\cdot}428 \text{ and } p_s = 0{\cdot}89\%$$

If the steel is less than 0·89%, then n is less than 0·428 and the section is weaker in tension than compression and will fail by tension.

We may take a range of values for n up to 0·428 and tabulate them as in Fig. 3.

The second column is merely $d\left(1 - \frac{n}{3}\right)$. The third column comes at once from equation (1) and the fifth column is obtained from equation (2a) by multiplying the second, third and fourth columns together.

We may next take a range of values for n from 0·428 upwards. These will correspond to values of p_s above 0·89%. The section will be stronger in tension than compression and will fail by the concrete reaching a stress of 750 lbs. per sq. in. These values are tabulated in Fig. 4. The sixth column is obtained by multiplying the fifth column by the lever arm. From Fig. 3 we see that as the percentage area of steel varies from 0·0308% to 0·89% the safe moment varies from 5·3 bd^2 to 138 bd^2. From Fig. 4 we see that as the percentage area of steel varies from 0·89% to 4·52% the safe moment varies from 138 bd^2 to 202 bd^2. Plotting these values of p_s against the values of R $\left(\text{which equals } \frac{M}{bd^2}\right)$ we have the curve, or rather two intersecting curves in Fig. 5. Notice that the safe moment increases rapidly and practically uniformly as p_s varies from 0

$m = 18$. Failure by $f_s = 18{,}000$ lbs. per sq. in.

n	LEVER ARM	p_s	f_s	SAFE M	R
0·1	0·967 d	0·0308%	18,000	5·3 bd^2	5·3
0·15	0·95 d	0·0735%	„	12·4 bd^2	12·4
0·2	0·933 d	0·139%	„	23·4 bd^2	23·4
0·25	0·917 d	0·232%	„	38·2 bd^2	38·2
0·3	0·90 d	0·358%	„	58 bd^2	58
0·35	0·883 d	0·522%	„	83 bd^2	83
0·4	0·867 d	0·74%	„	115 bd^2	115
0·428	0·857 d	0·89%	18,000	138 bd^2	138

Fig. 3.

to 0·89%. If we increase p_s beyond 0·89% the safe moment still increases but at a very much slower rate.

The reader must make himself familiar with figures such as Fig. 5 and he may draw similar figures for other values of f_c, f_s and m say 900, 16,000 and 14 or 1,000, 18,000 and 12, etc. There is no need to refer to any equations for the solutions which come easily from first principles.

$m = 18$. Failure by $f_c = 750$ lbs. per sq. in.

n	LEVER ARM	p_s	f_c	$\frac{1}{2}f_c\,b\,nd$	SAFE M	R
0·428	0·857 d	0·89%	750	161 bd	138 bd^2	138
0·45	0·85 d	1·02%	,,	169 bd	144 bd^2	144
0·5	0·833 d	1·39%	,,	188 bd	157 bd^2	157
0·55	0·817 d	1·87%	,,	216 bd	168 bd^2	168
0·6	0·80 d	2·50%	,,	225 bd	180 bd^2	180
0·65	0·783 d	3·36%	,,	244 bd	192 bd^2	192
0·7	0·767 d	4·52%	750	262 bd	202 bd^2	202

Fig. 4.

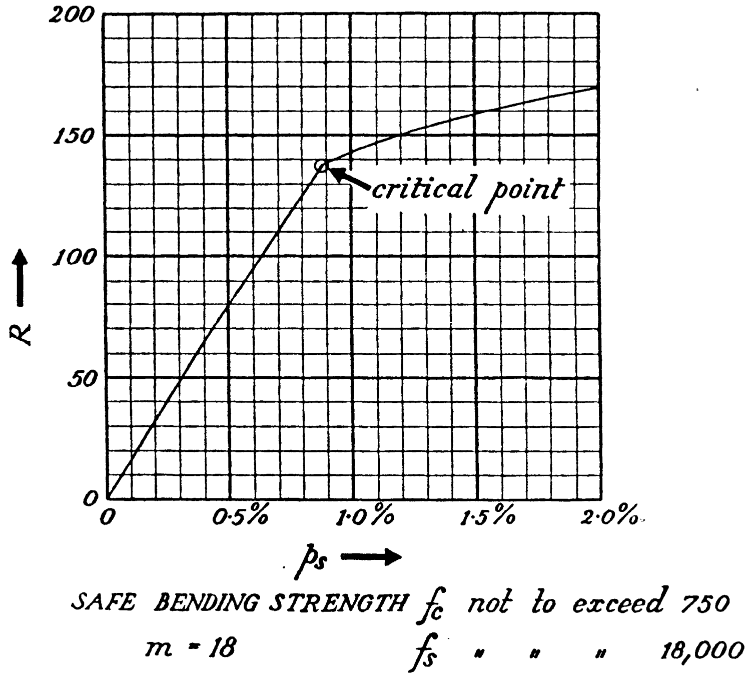


Fig. 5

Sections of Floor Slabs and Wall Slabs.—There are many considerations that enter into the matter but in standard practice nearly all floor and wall slabs are designed as rectangular beams reinforced in tension only and the sections of slabs come within the scope of this present chapter. Under present conditions it is best to aim at a section which has the critical values of n and p_s discussed above, that is, if we work to 750, 18,000 and $m = 18$ we aim at the point in Fig. 5 where the curve has a sudden corner, where $p_s = 0{\cdot}89\%$ and $M = 138\,bd^2$ (we may call this the " critical point " on the strength curve). A slight difficulty is introduced by the fact that practical slabs are made a whole number of half inches thick, that is 4″, 4½″, 5″, 5½″, 6″, 6½″, 7″, etc. We never attempt to make a slab 5⅛″ or $6\frac{17}{32}$″ or such fractions. Therefore we may in theory want a section which has an effective depth of, say, 4·37 ins. but in practice we should have to use a section 5½ ins. thick overall with a d of 4·75 ins. In general we have to take the nearest practical size larger than the depth calculated by theory. In other words, instead of exactly hitting the critical point in Fig. 5 we fall on the straight part of the strength curve just below the critical point and instead of requiring exactly 0·89% of steel we only require some smaller percentage. Yet even if this smaller percentage is only 0·80% (that is 10% below the 0·89% we aim at) the lever arm will be $0{\cdot}863\,d$ against $0{\cdot}857\,d$ for 0·89%—a difference of less than 1%. For all practical purposes the lever arm for all practical slab sections is constant (within less than 1%) and is equal to the lever arm at the critical percentage. This fact is used in design to find the amount of steel required from first principles and dispenses with the necessity of referring to any strength curves such as Fig. 5.

Suppose we have to find a section of a floor slab to resist a moment of 39,000 lbs.-ins. the section being 12 ins. wide, the allowable stresses being 750 and 18,000 with $m = 18$.

The critical point for these stresses occurs when $n = 0{\cdot}428$, lever arm $= 0{\cdot}857\,d$ and safe $M = 138\,bd^2$

$$M = 138\,bd^2$$

$$39{,}000 = 138 \times 12 \times d^2$$

$$\text{or } d = \sqrt{\frac{39{,}000}{138 \times 12}} = 4{\cdot}85''$$

Using ½-in. cover and ¼-in. bars, the theoretical overall depth

$$= 4{\cdot}85'' + \tfrac{1}{2}'' + \tfrac{1}{4}''$$

$$= 5{\cdot}60''$$

If we could use a slab 5·60 in. deep it would require exactly 0·89% of steel (on area bd).

$$\text{Area of steel } A_s = 0{\cdot}89\% \times 12'' \times 4{\cdot}85''$$

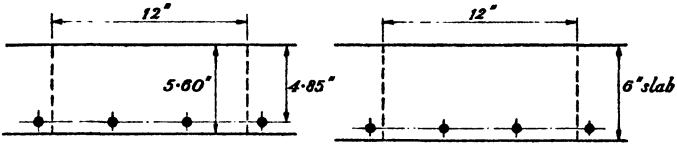
$$= 0{\cdot}519 \text{ sq. in.}$$

This theoretical section is shown on the left of Fig. 6. In practice we should use a 6-in. slab with $d = 5{\cdot}25$ ins. This will require less than 0·89% of steel. As the lever arm is very nearly 0·857 d, from first principles :—

$$A_s = \frac{39{,}000}{18{,}000 \times 0{\cdot}857 \times 5{\cdot}25} = 0{\cdot}482 \text{ sq. ins.}$$

We may use ½" bars every 5" (0·47 sq. ins.)
or ½" bars every 4½" (0·52 sq. ins.)

This section is shown on the right of Fig. 6.



A_s = 0·519 sq. ins.

Theoretical

½" bars at 4½" ctrs

Practical

Fig. 6

Bending Strength of a Given Section.—This problem very seldom occurs in practice. When it does, the curves marked $A_s' = 0$ in Tables RB I to RB VII in Chapter III may be used. Only in the case of examination questions is the problem likely to arise and one example must suffice.

A rectangular section is 10 ins. wide, 12 ins. deep overall, with 1 in. cover and is reinforced on the tension side with 4 bars 1 in. diam. If the maximum stress on the concrete is not to exceed 940 lbs. per sq. in., and on the steel 17,500 lbs. per sq. in., what is the maximum moment the section will resist ($m = 16{\cdot}5$) ?

$$b = 10'' \quad d = 10{\cdot}5''$$

$$A_s = 4 \text{ bars } 1'' = 3{\cdot}14 \text{ sq. ins.}$$

$$p_s = \frac{3{\cdot}14}{10 \times 10{\cdot}5} = 2{\cdot}99\%.$$

From equation (1)

$$2{\cdot}99 = \frac{50}{16{\cdot}5} \frac{n^2}{(1 - n)}$$

$$\text{giving } n = 0{\cdot}615$$

Moment of resistance if $f_c = 940$, by equation (2)

$$= \frac{1}{2} \times 940 \times 10 \times 0{\cdot}615 \times 10{\cdot}5\,(1 - 0{\cdot}205) \times 10{\cdot}5$$

= *254,000 lbs.–ins.*

Moment of resistance if $f_s = 17{,}500$ by equation (2a)

$$= \frac{2{\cdot}99 \times 10 \times 10{\cdot}5}{100} \times 17{,}500 \times (1 - 0{\cdot}205) \times 10{\cdot}5$$

= *459,000 lbs.–ins.*

Therefore although the steel will not reach a stress of 17,500 until the moment reaches 459,000 lbs.–ins., the concrete reaches a stress of 940 lbs. per sq. in. when the moment is 254,000 lbs.–ins. and this is the maximum safe moment the section will take.

To Design Sections for Slabs.—Most designers have standard stresses they adhere to. In any case it is unusual to vary the mix from one part of a structure to another, and once the critical point has been worked out, the values may be used throughout the structure. In the examples that follow a different set of stresses is taken for each which means calculating the critical values for each separate problem.

(1) Design a section of a floor slab 12 ins. wide to resist a moment of 45,000 lbs.–ins. Stresses of 600 and 16,000 with $m = 15$.

At critical point $f_c = 600$ and $f_s = 16{,}000$

$$\frac{f_s}{m} = \frac{16{,}000}{15} = 1066$$

$$n = \frac{600}{600 + 1066} = 0{\cdot}36$$

$$\text{lever arm} = \left(1 - \frac{0{\cdot}36}{3}\right)d = 0{\cdot}88\,d$$

$$\text{Compression on concrete} = \frac{1}{2} \times 600 \times b \times 0{\cdot}36\,d$$

$$\text{Moment of resistance} = \frac{1}{2} \times 600 \times b \times 0{\cdot}36\,d \times 0{\cdot}88\,d$$

$$= 95\,bd^2.$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{45{,}000}{12 \times 95}} = 6{\cdot}29 \text{ ins.}$$

Using ⅝-in. bars and ¾-in. cover use a slab 7½ ins. thick

actual d = say 6·4 ins.

$$A_s = \frac{45{,}000}{16{,}000 \times 0{\cdot}88 \times 6{\cdot}4} = 0{\cdot}5 \text{ sq. ins.}$$

say ⅝-in. bars every 7 in. (0·53).

(2) Design a section of a floor slab 12 ins. wide to resist a moment of 39,000 lbs.–ins. Stresses of 750 and 18,000 with $m = 18$.

At critical point $f_c = 750$ and $f_s = 18{,}000$.

$$\frac{f_s}{m} = \frac{18{,}000}{18} = 1000.$$

$$n = \frac{750}{750 + 1000} = 0{\cdot}428$$

$$\text{lever arm} = \left(1 - \frac{0{\cdot}428}{3}\right)d = 0{\cdot}857\,d$$

$$\text{Compression on concrete} = \frac{1}{2} \times 750 \times b \times 0{\cdot}428\, d$$

$$\text{Moment of resistance} = \frac{1}{2} \times 750 \times b \times 0{\cdot}428\, d \times 0{\cdot}857\, d$$

$$= 138\ bd^2$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{39{,}000}{12 \times 138}} = 4{\cdot}85 \text{ ins.}$$

using ½-in. bars and ½-in. cover use a slab 6 ins. thick

actual $d = 5{\cdot}25$ ins.

$$A_s = \frac{39{,}000}{18{,}000 \times 0{\cdot}857 \times 5{\cdot}25''} = 0{\cdot}482 \text{ sq. ins.}$$

say ½-in. bars every 4½ ins. (0·52).

(3) Design a section of a floor slab 12 ins. wide to resist a moment of 87,000 lbs.-ins. Stresses of 1000 and 17,000 lbs. per sq. in. with $m = 13$.

$$\text{At critical point } \frac{f_s}{m} = \frac{17000}{13} = 1310$$

$$n = \frac{1000}{1000 + 1310} = 0{\cdot}432$$

$$\text{lever arm} = \left(1 - \frac{0{\cdot}432}{3}\right)d = 0{\cdot}856\, d$$

$$\text{Moment of resistance} = \frac{1}{2} \times 1000 \times b \times 0{\cdot}432\, d \times 0{\cdot}856\, d$$

$$= 185\ bd^2$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{87{,}000}{12 \times 185}} = 6{\cdot}25 \text{ ins.}$$

Using ¾-in. bars and ¾-in. cover use a slab 7½ ins. thick.

actual d = say 6·25 ins.

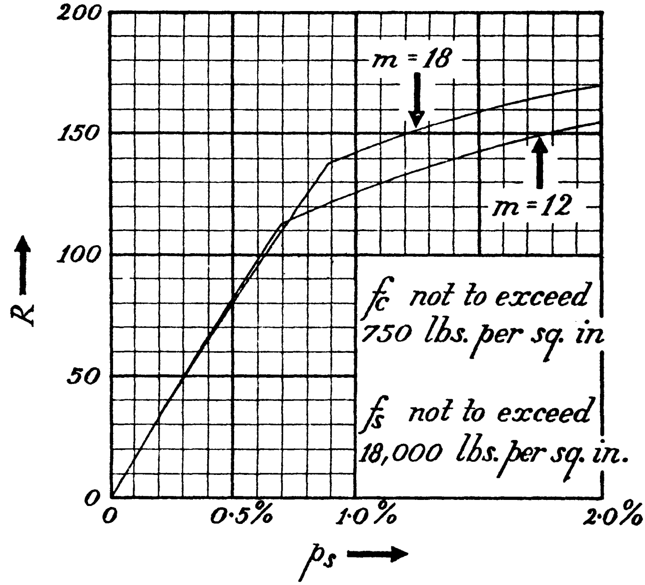
$$A_s = \frac{87{,}000}{17{,}000 \times 0{\cdot}856 \times 6{\cdot}25} = 0{\cdot}96 \text{ sq. ins.}$$

Say ¾-in. bars every 5½ ins. (0·96).

Maximum Strength.—Some designers have the idea that when working to stresses of, say, 750 and 18,000 with $m = 18$ that the value 138 bd^2 is the *maximum* moment that the section can resist without compression steel. A glance at Fig. 5 will show that this idea is wrong. It is, however, the most economical and most widely used value considering the construction of the floor as a whole. Some designers also think that the addition of more steel than the calculated value tends to weaken the section by throwing more stress on the concrete. It is clear from Fig. 5 that this idea is fallacious. Adding tensile steel always increases bending strength and reduces the stress on the concrete.

Cheapest Section.—The cheapest section, considering the slab by itself and not in relation to the other parts of the structure, depends on the relative costs of concrete and steel. If steel were very dear and concrete very cheap obviously a very deep section with a very small steel area is cheapest. If steel is very cheap and concrete very dear a shallow section with a very large percentage of steel is cheapest. In present normal cases the " critical point " should always be aimed at.

Effect of Varying *m*.—Since the value of m is so extremely vague and variable the question arises as to whether it is safest to assume a high or a low value for it. Fig. 7 shows two strength curves one for 750, 18,000 and $m = 18$, and one for 750, 18,000 and $m = 12$.



RELATIVE BENDING STRENGTHS

Fig. 7

In general the calculated strength, assuming $m = 18$, is higher than the calculated strength assuming $m = 12$. If $p_s = 0{\cdot}89\%$ assuming $m = 18$ gives $R = 138$, while assuming $m = 12$ gives $R =$ only 121.

Office Tables.—It is possible to construct a series of figures similar to Fig. 7, but these actually appear later as the lowest curves on Tables RB I—RB VIII.

It is possible to draw curves or tabulate values showing the critical values for all values of f_c, f_s and m, and to draw up tables showing the strength of slabs with various arrangements of bars. None of these are useful in practice as it is quicker to work out any values on the slide-rule than to find, read and apply an office table. The beginner should, however, make such a series of tables as the process will give valuable familiarity with the subject.

CHAPTER III

RECTANGULAR SECTIONS UNDER BENDING ONLY. GENERAL CASE

Analysis.

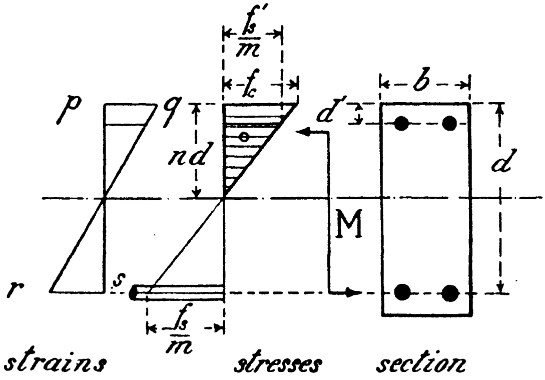


Fig. 8.

Let Fig. 8 represent a rectangular section subjected to a bending-moment M lbs.-ins.

breadth $= b$ inches
effective depth $= d$ inches

The section is reinforced with

A_s square inches of tensile steel.
A_s' square inches of compression steel.

We have by definition :—

$$\text{percentage of tensile steel} = p_s = \frac{A_s}{b \,.\, d} \times 100$$

$$\text{percentage of compression steel} = p_s' = \frac{A_s'}{b \,.\, d} \times 100.$$

This latter is placed d' from the compression edge.

Then, as in the last chapter :—

Let f_c represent the maximum fibre stress on the concrete.

f_s is the stress on the tensile steel
f_s' ,, ,, ,, ,, ,, compression steel

$pqrs$ represents the strains to some scale.

We have $\dfrac{\text{strain on tensile steel}}{\text{strain on concrete at edge}} = \dfrac{d - nd}{nd} = \dfrac{1 - n}{n}$

As the ratio of elastic moduli $= \dfrac{E_s}{E_c} = m$

$$\frac{\text{Stress on tensile steel}}{\text{Stress on concrete at extreme edge}} = \frac{f_s}{f_c} = \frac{m\,(1 - n)}{n}$$

$$\text{Similarly } \frac{f_s'}{f_c} = \frac{(nd - d')}{(nd)} \times m$$

Total compression on concrete

$$= \frac{1}{2} f_c\, bnd - (\text{stress on area occupied by } A_s').$$

Total compression on compression steel

$$= A_s' \times f_s' = A_s' \times \frac{(nd - d')}{nd} \times m f_c.$$

The addition of one square inch of compression steel to a section obviously displaces one square inch of concrete so that the nett increase of compressive resistance area is $(m - 1) \times$ area of steel in compression and

Total compression

$$= \frac{1}{2} f_c\, bnd + A_s' \times \frac{(nd - d')}{nd} \times (m - 1) f_c.$$

Total tension $= A_s \times f_s = A_s \times \dfrac{m\,(1 - n)}{n} f_c.$

Since the section resists a bending-moment only, these two must be equal.

i.e. $\dfrac{1}{2} f_c\, bnd + A_s' \times \dfrac{(nd - d')}{nd} \times (m - 1) f_c = A_s \times \dfrac{m\,(1 - n)}{n} f_c$

putting in the values $p_s = \dfrac{A_s}{b\,.\,d} \times 100$

$$p_s' = \frac{A_s'}{b\,.\,d} \times 100.$$

We have

$$\frac{100}{2} f_c\, bnd + p_s' \frac{(nd - d')}{nd}\,.\,b\,.\,d\,(m - 1) f_c = p_s bd\,.\,m \frac{(1 - n)}{n} f_c$$

$$50\, n + p_s' \frac{(nd - d')}{nd}\,.\,(m - 1) = p_s \frac{(1 - n)}{n}\,.\,m \quad . \; . \; . \; . \; . \; . \quad (3)$$

In order to determine n therefore we must know the values of

$$p_s,\ p_s',\ \frac{d'}{d} \text{ and } m.$$

Taking moments about the centre of the tensile steel and equating the resisting moment to the applied moment M, we have

$$\frac{1}{2}f_c bnd \times \left(d - \frac{nd}{3}\right) + A_s'(d - d').\frac{(nd - d')}{nd} \times (m-1)f_c = M$$

putting in the value of $p_s' = \frac{A_s'}{bd} \times 100$ and rearranging we get

$$\frac{1}{2}f_c n\left(1 - \frac{n}{3}\right) + \frac{1}{100}p_s'\left(1 - \frac{d'}{d}\right)\left(n - \frac{d'}{d}\right)\frac{(m-1)}{n}f_c = \frac{M}{bd^2}$$

which is the most useful form if f_c determines the strength.

$$\text{i.e. } M = bd^2 f_c\left[\frac{n}{2}\left(1 - \frac{n}{3}\right) + \frac{p_s'}{100}\left(1 - \frac{d'}{d}\right)\left(n - \frac{d'}{d}\right)\frac{(m-1)}{n}\right]. \quad (4)$$

Also if f_s is the deciding factor of strength we may write

$$M = bd^2 f_s\frac{n}{m(1-n)}\left[\frac{n}{2}\left(1 - \frac{n}{3}\right) + \frac{p_s'}{100}\left(1 - \frac{d'}{d}\right)\left(n - \frac{d'}{d}\right)\frac{(m-1)}{n}\right]. \quad (5)$$

The value of n can be obtained from equation (3) and substituted in equation (4) or equation (5).

It will be seen that these equations are incapable of being applied directly. They are however useful as they show one thing, namely, that, if two sections are similar, the moment of resistance varies directly as the breadth and as the square of the depth.

In fact if

(1) the value of m is the same for two sections.

(2) the value of p_s and p_s' is the same.

(3) the value of $\frac{d'}{d}$ is the same.

(4) The allowable values of f_c and f_s are the same.

Then the strength of the sections is relatively as bd^2.

Practical Application.—It is much the easiest course in practice to start from first principles and tabulate the result step by step.

We must assume a value for $\frac{d'}{d}$.

For simplicity we may assume $\frac{d'}{d} = \frac{1}{10}$.

This value is generally too small but the effect of the compression steel is in practice greater than the theory indicates (compare the "Steel-beam Theory").

We must next assume a value or series of values for m. The tables at the end of the chapter give readings for $m = 10, 12, 14, 15, 16$ and 18.

We must assume a value for $\frac{p_s'}{p_s}$ to reduce our unknowns. The tables show values of $\frac{p_s'}{p_s} = 0, = \frac{1}{2}, = 1.$

The curves are evenly spaced vertically and intermediate values as $\frac{p_s'}{p_s} = \frac{1}{4}$ or $= \frac{2}{3}$ are easily and safely interpolated by eye.

In general, compression steel is only added when there is danger of the concrete being overstressed and the tables are calibrated in terms of f_c. Values of f_s may be found by using the steel-stress lines shown on the tables.

Construction of Tables of Strength.

$$\text{Assume } n = 0{\cdot}45$$

$$\frac{d'}{d} = 0{\cdot}1$$

$$m = 14$$

$$\frac{p_s'}{p_s} = \tfrac{1}{2}$$

Following the lines of the general analysis (see Fig. 8)

$$\frac{f_s}{f_c} = \frac{14 \times 0{\cdot}55}{0{\cdot}45} \quad \text{i.e. } f_s = 17{\cdot}1\, f_c$$

$$\frac{f_s'}{f_c} = \frac{14 \times 0{\cdot}35}{0{\cdot}45} \quad \text{i.e. } f_s' = 14 \times \frac{0{\cdot}35}{0{\cdot}45} f_c$$

$$\text{actual } f_s' = 10{\cdot}9\, f_c$$

Since one square inch of compression steel replaces one square inch of concrete :—

$$\text{effective } f_s' = 13 \times \frac{0{\cdot}35}{0{\cdot}45} f_c = 10{\cdot}1\, f_c$$

$$\text{Total compression} = \frac{1}{2} f_c\, b \times 0{\cdot}45\, d + \frac{p_s'\, bd}{100} \times 10{\cdot}1 f_c$$

$$\text{Total tension} = \frac{p_s\, bd}{100} \times 17{\cdot}1\, f_c$$

Since these are equal and $p_s' = \frac{1}{2}\, p_s$ in this case

$$\frac{1}{2} f_c b \times 0{\cdot}45\, d + \frac{1}{2} p_s \frac{bd}{100} \times 10{\cdot}1\, f_c = \frac{p_s\, bd}{100} \times 17{\cdot}1 f_c$$

this gives $p_s = 1{\cdot}87\%$ and $p_s' = 0{\cdot}935\%$.

$$\text{Also } M = \frac{1}{2} f_c\, bnd^2 \left(1 - \frac{n}{3}\right) + p_s' (d - d') \times \frac{bd}{100} f_s' \times \frac{(m-1)}{m}$$

$$= \frac{1}{2} f_c b \times 0{\cdot}45\, d^2 (0{\cdot}85) + 0{\cdot}935 \times 0{\cdot}9 d \times \frac{bd}{100} \times 10{\cdot}9 f_c \times \frac{13}{14}$$

$$= 0{\cdot}191\, bd^2 f_c + 0{\cdot}085\, bd^2 f_c$$

$$= 0{\cdot}276\, bd^2 f_c$$

That is $\frac{R}{f_c} = 0{\cdot}276$ if $p_s = 1{\cdot}87\%$ and $p_s' = \frac{1}{2} p_s$

when $n = 0{\cdot}45$ and $f_s = 17{\cdot}1 f_c$

This gives one point on Table RB III.

The lines indicating the values of n are shown by dotted lines on Table RB III, but as we only use these lines as a means for calculating the value of f_s they have been marked "$f_s = 17{\cdot}1 f_c$" instead of "$n = 0{\cdot}45$," "$f_s = 14 f_c$" instead of "$n = 0{\cdot}5$" etc.

By taking other values for m and n and other ratios of $\frac{p_s'}{p_s}$ the whole of Tables RB I to RB VI may be plotted. The work may be simplified to a few arithmetical processes by tabulating the values many of which repeat.

Failure by Tension in Steel.—By using Tables RB I to RB VI we have the value of f_c directly but we only have the value of f_s indirectly by the ratio between f_s and f_c. When using values in the left-hand bottom corner of these tables, where this ratio is large, it is possible to overlook the fact that f_s may be excessive. On the other hand these tables are of universal application and may be used for any working stresses.

If we have a large number of structures all of which are to be designed to the same values of f_c, f_s and m it pays to make a special table like Table RB VII. This only applies to structures designed for $f_c = 750$, $f_s = 18{,}000$ and $m = 18$, but it shows the limits of strength both in tension and compression (compare Fig. 5) and gives the value of R directly. The reader should, as occasion arises, make such tables for himself. The steel-stress lines indicating the values of n can be shown to be straight lines which fact forms a useful guide in plotting values of R in such tables.

Checking Sections by Tables RB I to RB VI.—When the section is given we can at once calculate p_s and $\frac{A_s'}{A_s}$. The appropriate table gives the value of $\frac{M}{bd^2 f_c}$ and the ratio of f_s to f_c. If we know M we can calculate f_c, or if we know the maximum value of f_c we can find the maximum safe M. In both cases we can check f_s by using the value of the ratio f_s to f_c.

A section of a brace is as shown in Fig. 9.

If $m = 14$ and $M = 300{,}000$ lbs.-ins., what are the stresses f_c and f_s.

For $m = 14$ we use Table RB III.

$$A_s' = A_s = 2 \text{ bars } 1 \text{ in.} = 1{\cdot}57 \text{ sq. ins.}$$

$$bd = 120 \quad bd^2 = 1440$$

$$p_s = \frac{1{\cdot}57}{120} \times 100 = 1{\cdot}31\%$$

Taking the top curve $A_s' = A_s$ on Table RB III for a value $p_s = 1{\cdot}31\%$, we read

$$\frac{M}{bd^2 f_c} = 0{\cdot}275$$

and $f_s = 24 f_c$ (by interpolation).

$$\text{Therefore} \quad \frac{300{,}000}{1440\, f_c} = 0{\cdot}275$$

giving $f_c =$ *760 lbs. per sq. in.*

$f_s = 24 f_c$ or $f_s =$ *18,200 lbs. per sq. in.*

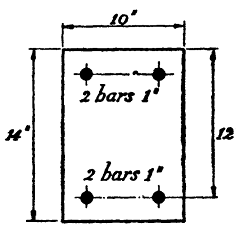


Fig. 9

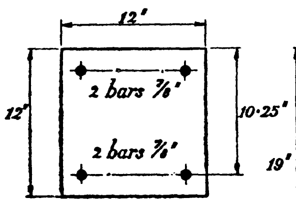


Fig. 10

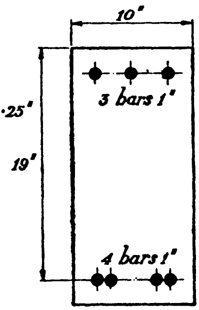


Fig. 11

A pile is 12 ins. × 12 ins. in section reinforced with four bars $\frac{7}{8}$-in. diam. as in Fig. 10. While being lifted it is subjected to a moment of 200,000 lbs.-ins. If $m = 15$ what stresses are caused?

$$b = 12''$$

$$d = 10{\cdot}25''$$

$$A_s' = A_s = 2 \text{ bars } \tfrac{7}{8}\text{-in.} = 1{\cdot}2 \text{ sq. ins.}$$

$$\therefore p_s = \frac{1{\cdot}2}{12 \times 10{\cdot}25} = 0{\cdot}975\%.$$

$$\text{From Table RB IV,} \quad \frac{R}{f_c} = 0{\cdot}24$$

$$\text{and} \quad f_s = \text{about } 28 f_c$$

$$R = \frac{200{,}000}{12 \times 10{\cdot}25^2} = 158$$

$$f_c = \frac{158}{0{\cdot}24} = 660 \text{ lbs. per sq. in.}$$

$$f_s = \text{about } 28 \times 660 = 18{,}500 \text{ lbs. per sq. in.}$$

The following example illustrates the necessity of checking f_s particularly when working to unusual stresses.

A section is 10 ins. × 19 ins. effective depth as in Fig. 11, and is reinforced with 4 bars 1 in. in tension and 3 bars 1 in. in compression.

What is its maximum safe bending strength if f_c must not exceed 1,000 and f_s must not exceed 15,000 lbs. per sq. in. with $m = 12$.

$$A_s = 4 \text{ bars } 1 \text{ in.} = 3{\cdot}14 \text{ sq. ins.}$$

$$p_s = \frac{3{\cdot}14}{10 \times 19} = 1{\cdot}65\ \%$$

$$A_s' = \tfrac{3}{4} A_s$$

Turning to Table RB II and drawing an imaginary line halfway between the curves $A_s' = A_s$ and $A_s' = \frac{1}{2} A_s$, we have

$$\frac{R}{f_c} = 0{\cdot}265$$

$$f_s = \text{say } 18 f_c$$

If $f_c = 1{,}000$ then $R = 265$

$$\text{and} \quad M = 265 \times 10 \times 19^2$$

$$= \mathit{958{,}000\ lbs.\text{-}ins.}$$

but if $f_c = 1{,}000$ and $f_s = 18\ f_c$ then $f_s = 18{,}000$, which is much more than the allowable value.

If f_s is not to exceed 15,000 and $f_s = 18 f_c$, then f_c must not exceed $\frac{15{,}000}{18}$ or 833 lbs. per sq. in.

$$\text{and as } R = 0{\cdot}265\, f_c$$

$$R = 0{\cdot}265 \times 833$$

$$\text{and} \quad M = 0{\cdot}265 \times 833 \times 10 \times 19^2 = \mathit{800{,}000\ lbs.\text{-}ins.}$$

Therefore M cannot exceed 800,000 lbs.-ins. or the steel will be overstressed.

Designing Sections by Tables RB I to RB VI.—Many rectangular beam sections are not really designed—they merely occur. The values of b and d are determined by factors other than the bending moment at the section analysed as are also in many cases the values of A_s and A_s'. For example, as will be seen later, a T-beam in a floor becomes a rectangular beam under reversed moment. The main dimensions of the beam are determined by the moment at mid-span, the number and arrangement of bars and the shear at the support. Thus the section at the support is largely determined by the conditions existing elsewhere and all that can be done is to check the section thus arrived at and make such minor alterations as are necessary.

The sections of isolated bracings are dependent on the ratio of depth to span and the ratio of breadth to span. These can only be fixed when we come to look at the member as a whole (see Chapter XIII).

As a rough guide the breadth of an isolated beam is usually about half the depth. For a light brace 1% of steel top and bottom. For medium or heavy braces 2% to 3% of steel top and bottom (that is 4% to 6% total).

While it is easy to check a given section, to design a rectangular section to resist a given moment is difficult because of the very large number of possible solutions. We may use a small section with a large amount of steel or a large section with a small amount of steel. We may use a broad shallow section or a deep narrow one. We may make A_s' equal to A_s or equal to zero or any intermediate value. For example, all the six sections in Fig. 12 have a safe bending strength of 800,000 lbs.-ins. working to stresses of 750 and 18,000 with $m = 18$. Dozens of further alternatives could be found with other values of b and d and intermediate ratios of A_s' to A_s. Which one of all these alternatives is best depends on the special circumstances attaching to any particular case.

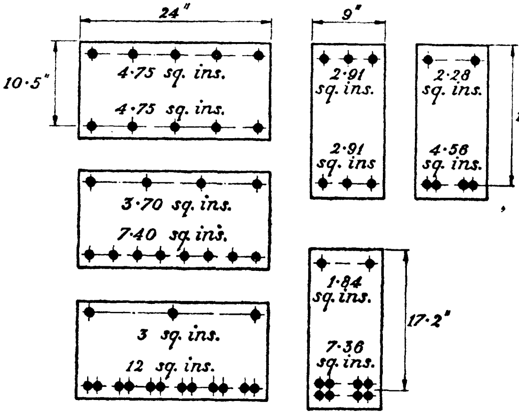


Fig. 12

The use of the curves in the tables automatically ensures that f_c will not exceed the allowable value. But we must also keep to the right of the appropriate steel-stress line to ensure that f_s as well will not exceed its allowable value. If for some particular reason we have a case which falls close to the left-hand bottom corner of a table it is obvious that no compression steel is called for and the section may be treated by the methods in Chapter II.

Examples.—A brace is 10 ins. wide and 16·5 ins. effective depth. It has to resist a moment of 1,200,000 lbs.-ins. or a reversed moment of 1,200,000 lbs.-ins. If f_c is not to exceed 900 lbs. per sq. in. and f_s is not to exceed 24,000 lbs. per sq. in., what steel is required if $m = 18$?

As the moment may be either positive or negative, we shall use the same area of steel top and bottom, that is $A_s' = A_s$.

$$R = \frac{1{,}200{,}000}{10 \times 16{\cdot}5^2} = 440.$$

$$\text{If} \quad f_c = 900 \text{ then } \frac{R}{f_c} = 0{\cdot}49.$$

The ratio $\frac{f_s}{f_c}$ if f_s is 24,000 and f_c is 900 is 26·7.

If we pick a value to the right of the line

$$f_s = 26{\cdot}7\, f_c \text{ then } f_s \text{ will be safe.}$$

Referring to Table RB VI if $A_s' = A_s$ with $p_s = 2{\cdot}5\%$

$$\text{then } \frac{R}{f_c} = 0{\cdot}49 \text{ and } f_s = \text{about } 22\, f_c.$$

Therefore if we use a section with 2½% of steel top and bottom, f_c will be 900

and f_s about 22 × 900 or 19,800 lbs. per sq. in. which is safe.

$$2\tfrac{1}{2}\% \text{ on } 10'' \times 16{\cdot}5'' = 4{\cdot}11 \text{ sq. ins.}$$

In practice say 4 bars 1⅛ ins. top and bottom (see Fig. 13).

Design a medium section to resist a moment of 600,000 lbs.–ins. with stresses not to exceed 750 and 18,000 and $m = 18$. Make the section about half as wide as it is deep. In Table RB VII we have a direct-reading table for these stresses. With 2% top and bottom (that is $A_s' = A_s$ and $p_s = 2\%$) $R = 315$

$$\frac{600{,}000}{bd^2} = 215$$

If $b = \frac{1}{2} d$ then $\frac{600{,}000}{\frac{1}{2}d^3} = 315$

$$d^3 = 3{,}810$$
$$d = 15{\cdot}6$$

A section 8 ins. × 18 ins. overall is therefore about right with d equal to about 15·5.

$$\text{Actual } bd^2 = 8 \times 15{\cdot}5^2$$
$$= 1930$$
$$\text{Actual } R = \frac{600{,}000}{1930} = 310.$$

From Table RB VII $A_s' = A_s$ with $p_s = 1{\cdot}95\%$.

$$= \frac{1{\cdot}95 \times 8 \times 15{\cdot}5}{100} = 2{\cdot}42 \text{ sq. ins.}$$

Steel say 4 bars ⅞–in. (2·4 sq. ins.) top and bottom as in Fig. 14.

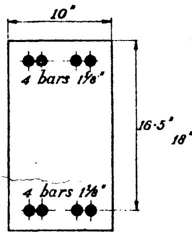


Fig. 13.

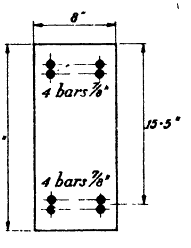


Fig. 14.

The following example illustrates the necessity of referring to the steel-stress lines. A section is 10 ins. × 16 ins. effective depth. It has to resist a moment of 770,000 lbs.–ins. which may act in either sense sometimes causing tension on the top edge of the section, sometimes on the bottom. Stresses not to exceed 900 and 16,000 lbs. per sq. in. with $m = 15$.

$$R = \frac{770{,}000}{10 \times 16^2} = 300$$

$$\frac{R}{f_c} = \frac{300}{900} = 0{\cdot}33$$

The limiting steel stress ratio is $\dfrac{f_s}{f_c} = \dfrac{16{,}000}{900} = 17{\cdot}8$.

Since the moment may act in either sense we want a section having the same steel top and bottom. Referring to Table RB IV, the line $\dfrac{R}{f_c} = 0{\cdot}33$ cuts the line $A_s' = A_s$ at the point where p_s is 1·7%, but this point is well to the left of the steel-stress line $f_s = 17{\cdot}8\,f_c$. It is in fact on the line $f_s = 22{\cdot}5\,f_c$, and if we use 1·7% of steel top and bottom f_c will be only 900 but f_s will be

$$900 \times 22{\cdot}5 = 20{,}300 \text{ lbs. per sq. in.}$$

If we take the point where the steel-stress line $f_s = 17{\cdot}8\,f_c$ cuts the line $A_s' = A_s$ we find $p_s = 3{\cdot}35\%$ and $\dfrac{R}{f_c} = 0{\cdot}509$.

Since $R = 300$ this gives $f_c = 592$ lbs. per sq. in., and as we are on the line $f_s = 17{\cdot}8\,f_c$

$$\text{then } f_s = 17{\cdot}8 \times 592 = 10{,}600 \text{ lbs. per sq. in.}$$

If we use 3·35% of steel top and bottom, the stresses are far too low.

If we take the point where the value $\dfrac{R}{f_c} = 0{\cdot}33$ cuts the steel-stress line $f_s = 17{\cdot}8 f_c$ we have

$$p_s = 2{\cdot}2\% \text{ and } A_s' \text{ about } 0{\cdot}65\,A_s \text{ say } 1{\cdot}43\%.$$

Such a section has stresses of exactly 900 and 16,000, but is not symmetrical, yet it is, in fact, the solution of the problem. A section with 2·2% of steel at the bottom and 1·43% at the top will resist a positive moment of 770,000 lbs.–ins., while a section with 2·2% at the top and 1·43% at the bottom will resist a negative moment of 770,000 lbs.–ins. Therefore a beam with 2·2% of steel top and bottom will resist both.

$$2{\cdot}2\% = \frac{2{\cdot}2 \times 10 \times 16}{100} = 3{\cdot}53 \text{ sq. ins. top and bottom.}$$

(See Fig. 15.)

We may check this back.

$$\text{If } A_s' = A_s = 2{\cdot}2\% \text{ then } \frac{R}{f_c} = 0{\cdot}386$$

$$\text{and } f_s = \text{about } 20{\cdot}5 f_c$$

$$\text{As } R = 300 \quad f_c = \frac{300}{0{\cdot}386} = \textit{780 lbs. per sq. in.}$$

$$f_s = 20{\cdot}5 \times 780 = \textit{16,000 lbs, per sq. in.}$$

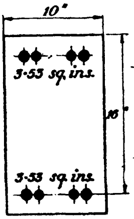


Fig. 15.

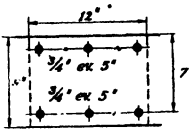


Fig. 16.

As an example of a section that appears to come within the scope of this chapter but which really does not, we may take the section in Fig. 16. $b = 12$ ins. and $d = 7\frac{3}{4}$ ins. with allowable stresses of 900 and 16,000 and $m = 14$. The section must resist a moment of 115,000 lbs.-ins. in either sense.

$$R = \frac{115{,}000}{12 \times 7{\cdot}75^2} = 160$$

$$\frac{R}{f_c} = 0{\cdot}178 \text{ and } \frac{f_s}{f_c} = \frac{16{,}000}{900} = 17{\cdot}8$$

If we trace these two lines on Table RB III we find that they intersect *below* the line $A_s' = 0$, a point which is not really on the table at all as A_s' cannot be *less* than zero. The case really falls in Chapter II for with stresses of 900 and 16,000 with $m = 14$, the " critical " value of R is 170 with lever arm $0{\cdot}853\,d$ and no compression steel at all is required when R is less than 170. Following the methods in Chapter II

$$A_s = \frac{115{,}000}{16{,}000 \times 0{\cdot}853 \times 7{\cdot}75} = 1{\cdot}08 \text{ sq. ins.}$$

We require 1·08 sq. ins. at the bottom to resist a positive moment of 115,000 lbs.-ins. and 1·08 sq. ins. at the top to resist a negative moment of 115,000 lbs.-ins.

The " Steel-Beam " Theory.—It has been put forward that a beam having equal steel top and bottom such as shown in Fig. 17 may be treated as a steel beam and that its strength may be calculated on the assumption that the concrete area can be neglected as far as bending stresses are concerned.

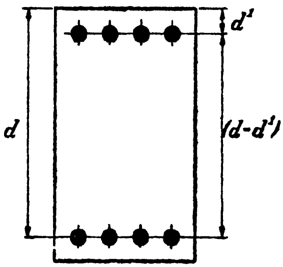


Fig. 17.

Working to a stress of 18,000 lbs. per sq. in. on the steel in tension and compression the bending strength of such a section is

$$M = 18{,}000 \times A_s (d - d')$$

There can be no doubt that this contention is wrong in principle and that the idea, when pushed to its logical (?) conclusion, leads to cramped sections overcrowded with steel surrounded with badly consolidated concrete full of air pockets, weak in shear and shear-bond strength. Many continental designers (quite rightly) refuse to accept the principle.

As a rough mental check on a section by an experienced designer, the idea may save time, but in the hands of a beginner, or one who has no outside experience, it is a dangerous doctrine.

As a comparison between the strength calculated by the " steel-beam " theory and by the standard method the figures in Fig. 18 furnish a typical example.

CALCULATED STRENGTH OF RECTANGULAR SECTION WITH SAME STEEL TOP AND BOTTOM ON DIFFERENT ASSUMPTIONS

	STEEL-BEAM THEORY	STANDARD THEORY		
		$m = 10$	$m = 14$	$m = 18$
$A_s' = A_s = 1\%$	171 bd^2	146 bd^2	162 bd^2	165 bd^2
$= 2\%$	342	216	265	313
$= 3\%$	513	278	347	416
$= 4\%$	682	334	422	513

Fig. 18.

Assuming that $d' = 0{\cdot}1\,d$ in all cases and that f_s is not to exceed 18,000 and f_c not to exceed 750 lbs. per sq. in., the calculated strength of the section is given first for the " steel-beam " theory, then on the standard theory for $m = 10$, $m = 14$ and $m = 18$.

The strength by the "steel-beam" theory is always greater than that calculated by the standard theory. The higher the percentage of steel and the lower the assumed value of m the greater the difference. With 4% top and bottom the strength given by the steel-beam theory is more than double the strength given by the standard theory assuming $m = 10$.

The fact that the steel-beam theory is very easy to apply appeals strongly to the beginner. The fact that it gives a very high calculated strength appeals strongly to those who engage in cut-throat competitive design. The author would limit its application to beams with 3% of steel top and bottom or less.

Office Tables.—Most of the author's office tables are about foolscap size. The placing of bars is not nearly so accurate as might be imagined and interpolation by eye on these tables is quite as accurate as circumstances warrant.

It is fatally easy in drawing up tables to overcrowd them with lines to such an extent that it is almost impossible to read them. It is much the wiser plan to spread the information over two or three tables and omit as far as possible all non-essential lines which may add to the interest of the diagram and yet destroy its usefulness. All the vital assumptions on which the curves are based *must be written on the tables themselves.*

The working out and tabulation of values gives great familiarity with the subject and readers are earnestly advised to draw out their own office tables and curves.

Assumed Value of m.—It is clear from a comparison of Tables RB I to RB VI that for given values of f_c and f_s the higher the assumed value of m the greater is the calculated bending strength. As the effective value of m is so vague and so variable we may rely on the fact that most existing structures have been designed on the assumption that m is 15 and we cannot go far wrong if we take some such value.

Dimensions of R.—Although R is generally quoted as a pure number, e.g. $R = 138$ when $M = 138\, bd^2$, this is not strictly accurate.

As $M = R\, bd^2$ the dimensions of the two sides of the equation must be equal. If M is in pounds-inches and b and d are in inches then R must have the dimensions of pounds per square inch.

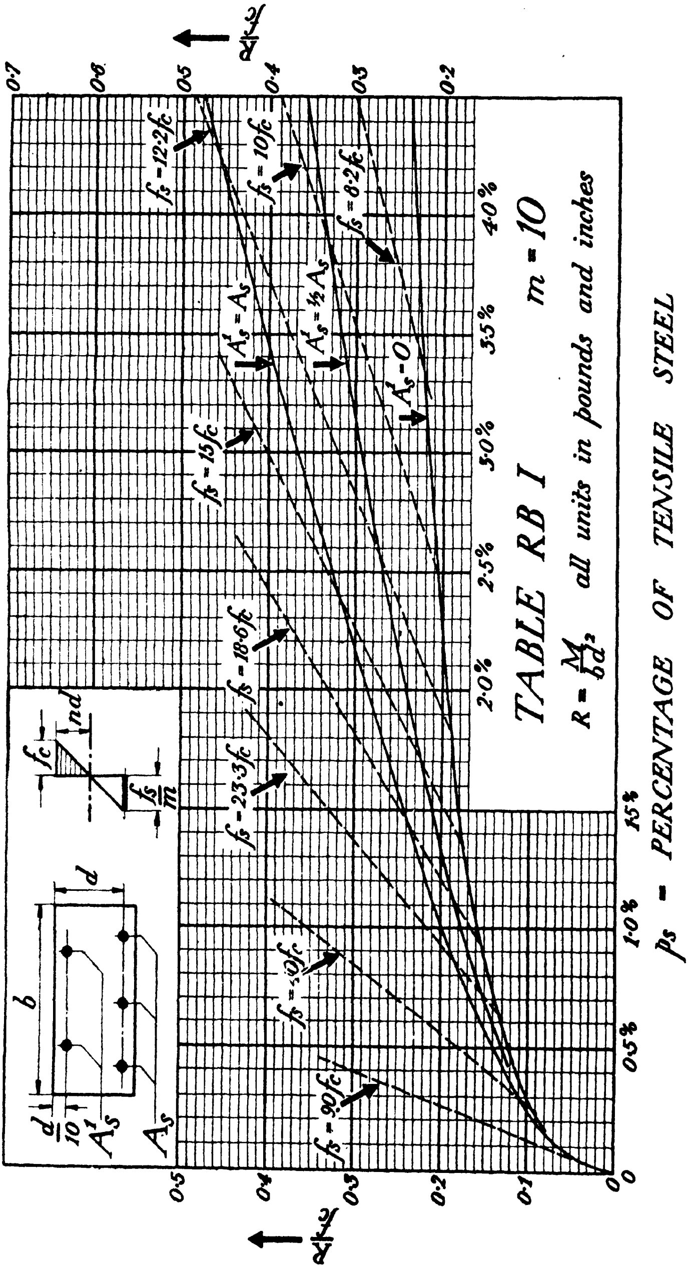


TABLE RB I
m = 10
R = M/bd²
all units in pounds and inches
p_s - PERCENTAGE OF TENSILE STEEL
0
0·5%
1·0%
1·5%
2·0%
2·5%
3·0%
3·5%
4·0%
R/fc
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
$f_s = 90 f_c$
$f_s = 50 f_c$
$f_s = 23{\cdot}3 f_c$
$f_s = 18{\cdot}6 f_c$
$f_s = 15 f_c$
$f_s = 12{\cdot}2 f_c$
$f_s = 10 f_c$
$f_s = 8{\cdot}2 f_c$
$A_s^1 = A_s$
$A_s^1 = \frac{1}{2} A_s$
$A_s^1 = 0$
d/10
A_s^1
A_s
b
d
f_c
nd
f_s/m

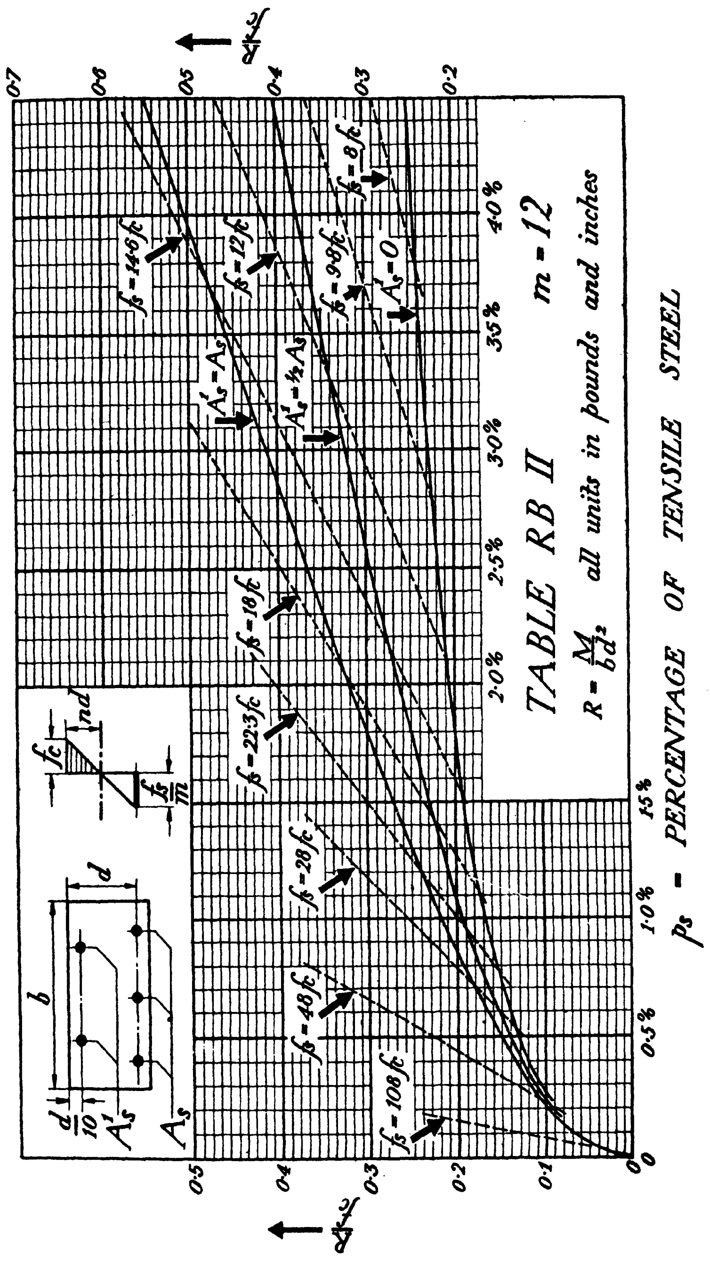


TABLE RB II
m = 12
R = M/bd²
all units in pounds and inches
ps = PERCENTAGE OF TENSILE STEEL
R/fc
R/fc
0·7
0·6
0·5
0·4
0·3
0·2
0·5
0·4
0·3
0·2
0·1
0
0
0·5%
1·0%
1·5%
2·0%
2·5%
3·0%
3·5%
4·0%
fs = 14·6fc
fs = 12fc
fs = 9·8fc
fs = 8fc
A's = As
A's = ½As
A's = 0
fs = 18fc
fs = 22·3fc
fs = 28fc
fs = 48fc
fs = 108fc
d/10
A's
As
b
d
fc
nd
fs/m

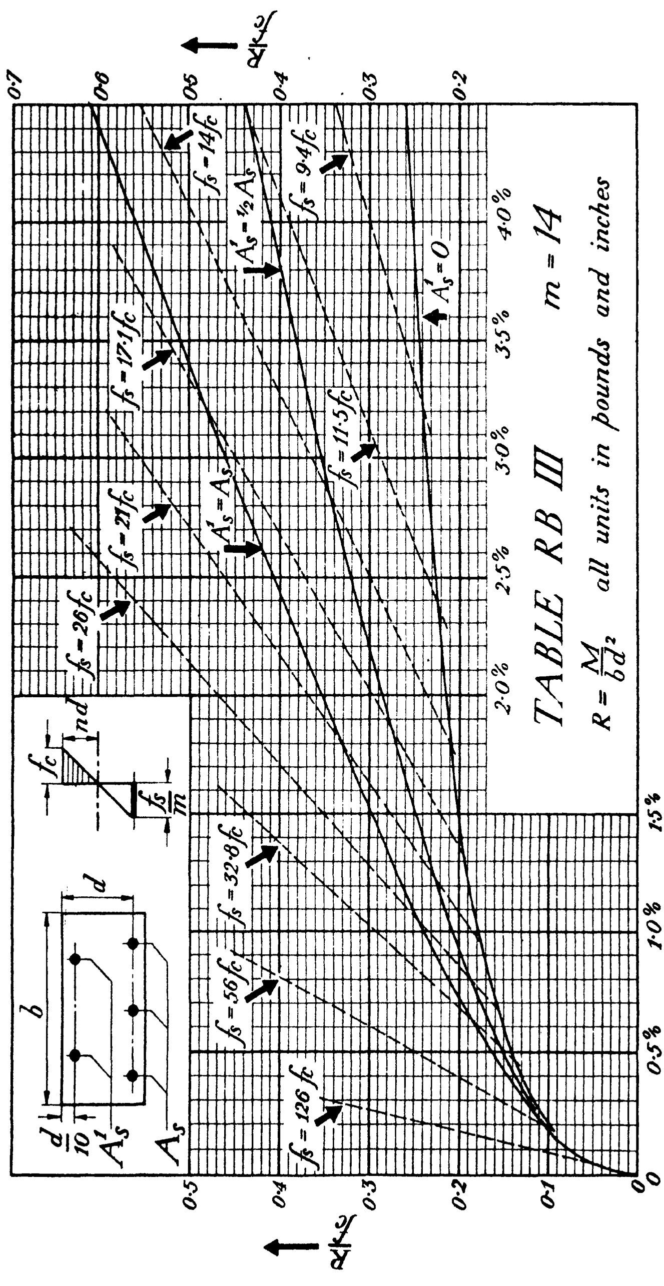


TABLE RB III
m = 14
R = M/bd²
all units in pounds and inches
ps = PERCENTAGE OF TENSILE STEEL
R/fc
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0
0·5%
1·0%
1·5%
2·0%
2·5%
3·0%
3·5%
4·0%
fs = 126 fc
fs = 56 fc
fs = 32·8 fc
fs = 26 fc
fs = 21 fc
fs = 17·1 fc
fs = 14 fc
fs = 11·5 fc
fs = 9·4 fc
A's = As
A's = ½ As
A's = 0
b
d
d/10
A's
As
fc
nd
fs/m

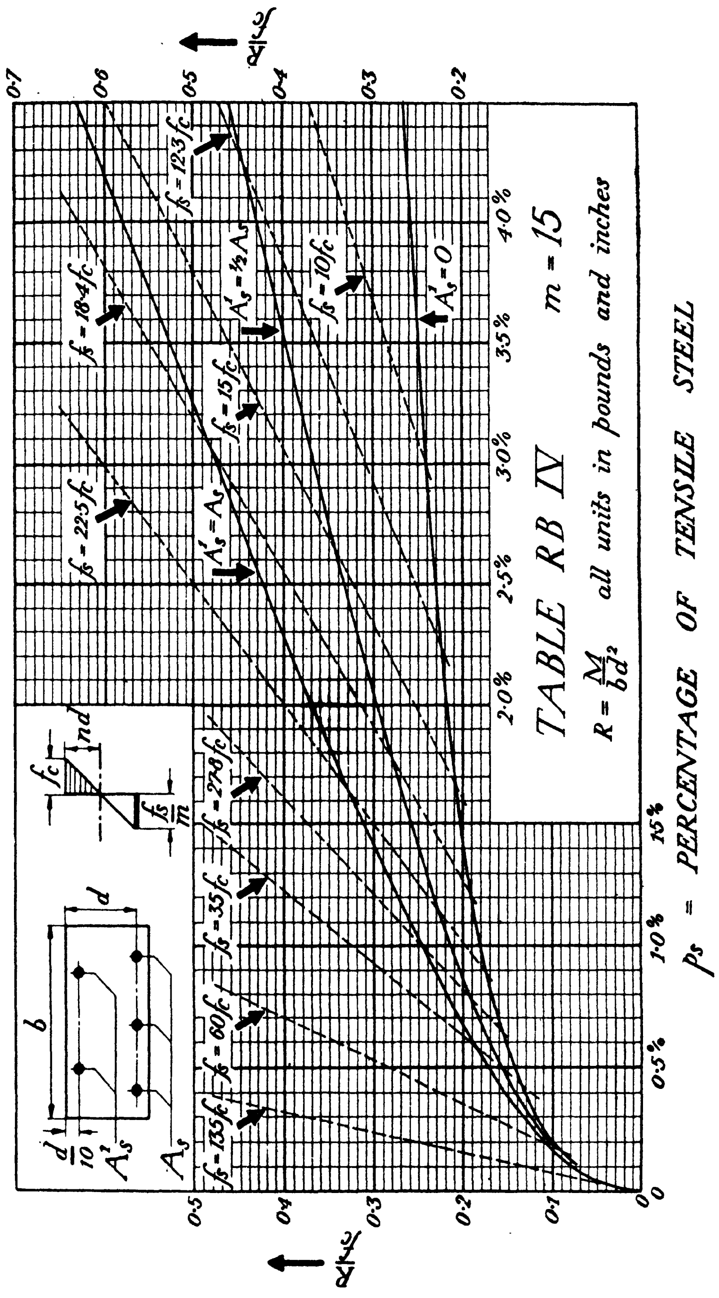


TABLE RB IX
$m = 15$
$R = \frac{M}{bd^2}$ all units in pounds and inches
p_s = PERCENTAGE OF TENSILE STEEL
$\frac{R}{f_c}$
$\frac{R}{f_c}$
0·7
0·6
0·5
0·4
0·3
0·2
0·5
0·4
0·3
0·2
0·1
0
0
0·5%
1·0%
1·5%
2·0%
2·5%
3·0%
3·5%
4·0%
$f_s = 135 f_c$
$f_s = 60 f_c$
$f_s = 35 f_c$
$f_s = 27{\cdot}8 f_c$
$f_s = 22{\cdot}5 f_c$
$f_s = 18{\cdot}4 f_c$
$f_s = 15 f_c$
$f_s = 12{\cdot}3 f_c$
$f_s = 10 f_c$
$A'_s = A_s$
$A'_s = \frac{1}{2} A_s$
$A'_s = 0$
b
d
$\frac{d}{10}$
A'_s
A_s
f_c
nd
$\frac{f_s}{m}$

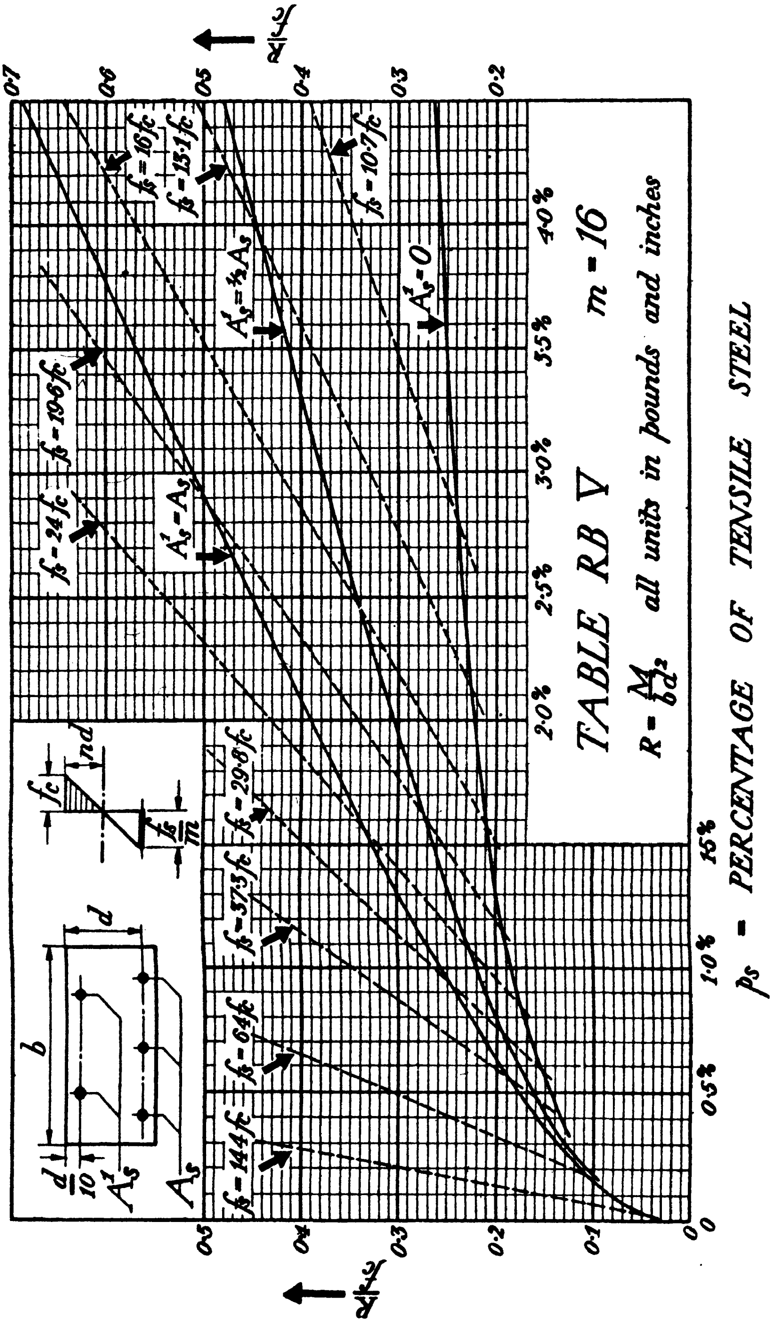


TABLE RB V

$R = \frac{M}{bd^2}$ all units in pounds and inches

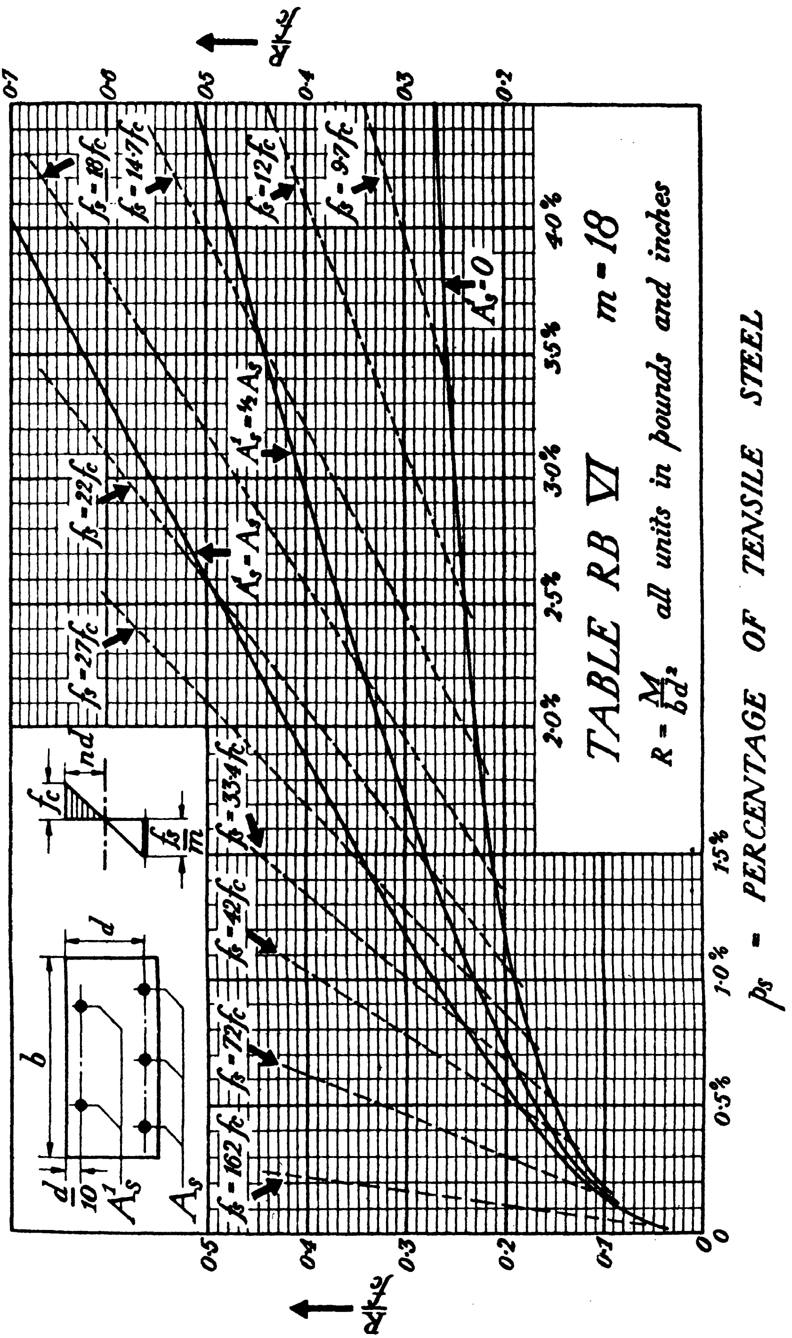


TABLE RB VI m = 18
R = M/bd²
all units in pounds and inches
ps = PERCENTAGE OF TENSILE STEEL
R/fc
R/fc
0
0·5%
1·0%
1·5%
2·0%
2·5%
3·0%
3·5%
4·0%
0·1
0·2
0·3
0·4
0·5
0·6
0·7
fs = 162 fc
fs = 72 fc
fs = 42 fc
fs = 33·4 fc
fs = 27 fc
fs = 22 fc
fs = 18 fc
fs = 14·7 fc
fs = 12 fc
fs = 9·7 fc
A¹s = As
A¹s = ½ As
A¹s = 0
b
d
d/10
A¹s
As
fc
nd
fs/m

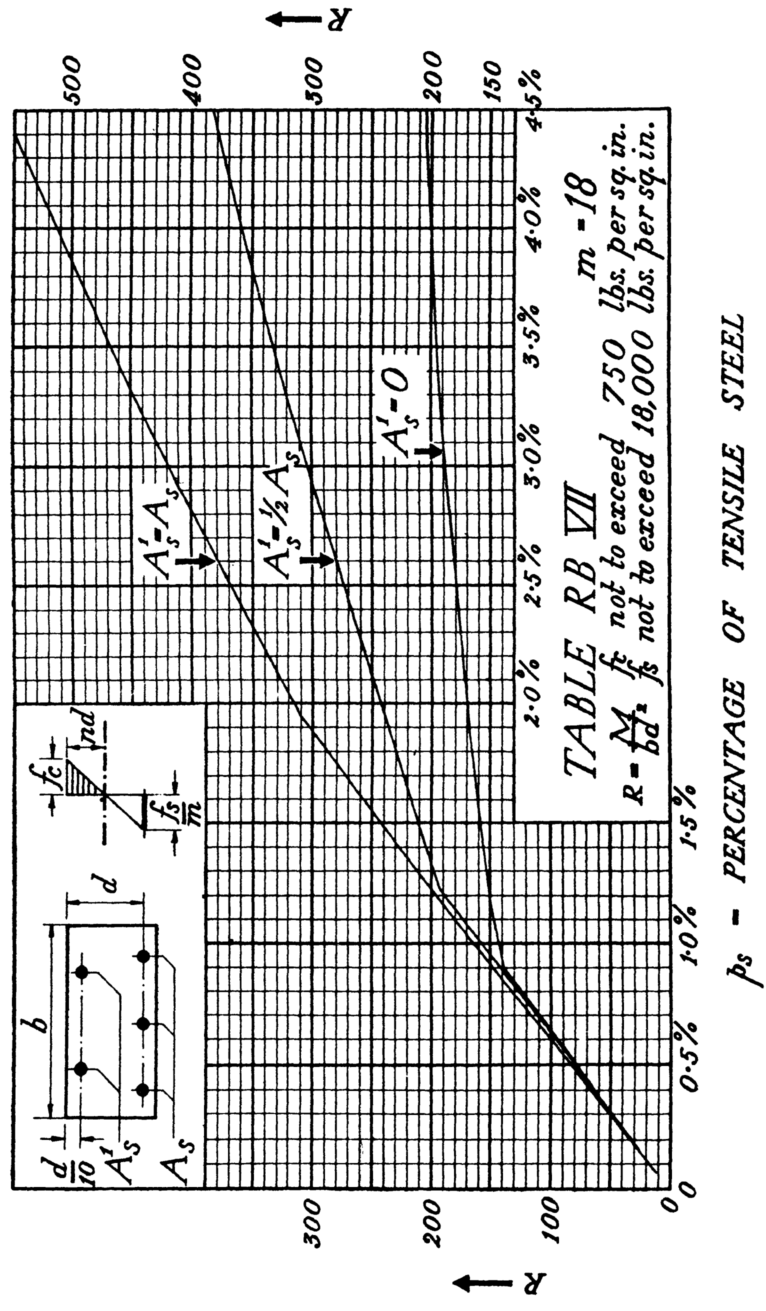


TABLE RB VII
m = 18
$R = \frac{M}{bd^2}$
f_c not to exceed 750 lbs. per sq. in.
f_s not to exceed 18,000 lbs. per sq. in.
$A'_s = A_s$
$A'_s = \frac{1}{2} A_s$
$A'_s = 0$
p_s - PERCENTAGE OF TENSILE STEEL
R
500
400
300
200
150
100
0
0
0·5%
1·0%
1·5%
2·0%
2·5%
3·0%
3·5%
4·0%
4·5%
b
d
$\frac{d}{10}$
A'_s
A_s
f_c
nd
$\frac{f_s}{m}$

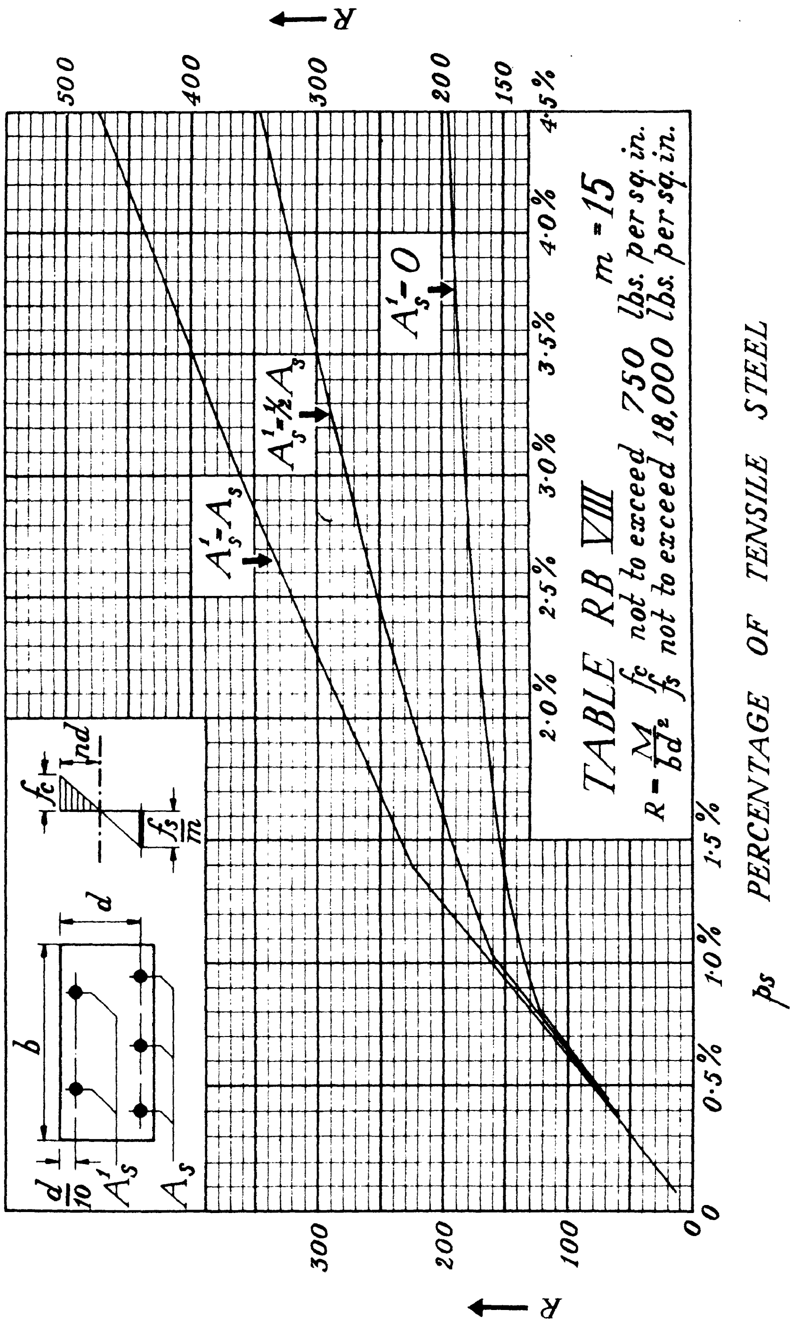


TABLE RB VIII $m = 15$

$R = \frac{M}{bd^2}$ f_c not to exceed 750 lbs. per sq. in.

f_s not to exceed 18,000 lbs. per sq. in.

L.C.C. Regulations.—The regulations at present in force (1938) allow stresses of 750 and 18,000 on 1 : 2 : 4 concrete with $m = 15$. These give a critical point with $n = 0{\cdot}385$, $p_s = 0{\cdot}803\%$ and $R = 126$. The bending strength of sections with compression steel is given in Table RB VIII which should be compared with Table RB VII.

CHAPTER IV

T-SECTIONS UNDER BENDING ONLY

Analysis.

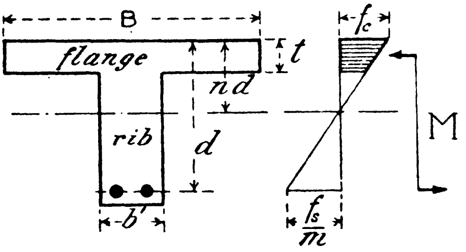


Fig. 19.

The left-hand part of Fig. 19 represents a T-section subjected to a bending-moment M. The right-hand part represents the stresses produced in the material.

Let nd = depth of neutral axis from the compressed edge

f_c = maximum compressive stress in concrete

f_s = tensile stress in steel

m = modular ratio $= \frac{E_s}{E_c}$.

The reason for employing such a section is self-evident. The flange of concrete supplies resistance to compression while the vertical stalk of the T gives depth and lever arm. Steel will safely carry about 25 times as much stress as the same area of concrete and moreover the whole of the tensile steel carries the same stress while the stress on the concrete rapidly diminishes as one approaches the neutral axis. A small width b' will contain enough area of steel to prevent failure of the section by tension.

There are, however, other important considerations which fix the values of b' and d which will appear later.

(This is one of the points which make reinforced concrete design so difficult. In carrying through a design the engineer cannot definitely settle the section of a member at any point until he has designed the whole member and in some cases has roughly designed the members which carry it also. For example in the calculation of a slab-and-girder floor it is not possible to design the floor-slab without having some idea of the moments carried by the secondary and main beams since the floor slab not only acts as flooring but also actually constitutes an integral part of all the beams. Practice, of course, will help the designer to make a brief mental survey of the whole structure before commencing to calculate the details of the separate members.)

Following the methods of Chapters II and III:

Maximum stress on concrete at top edge of flange $= f_c$.

Stress at lower side of flange

$$= f_c \times \frac{nd - t}{nd}.$$

Tensile stress on steel

$$= f_s = m \times f_c \times \frac{d - nd}{nd} = f_c \times \frac{m\,(1 - n)}{n}.$$

Total stress on concrete flange

$$= \text{area} \times \text{average stress}$$

$$= Bt \times \frac{1}{2}\left(f_c + f_c \times \frac{(nd - t)}{nd}\right).$$

Total compression on rib

$$= b' \times \frac{1}{2} f_c \times \frac{(nd - t)}{nd} \times (nd - t).$$

Total tension on steel

$$= A_s \times f_s$$

$$= A_s \times f_c \times \frac{m\,(1 - n)}{n}.$$

Since the section resists a bending moment only, the total tension equals the total compression $A_s \times f_c \times \dfrac{m\,(1 - n)}{n}$

$$= \frac{Bt}{2}\left(f_c + f_c \frac{(nd - t}{nd}\right) + \frac{b'}{2}\left(f_c \times \frac{(nd - t)^2}{nd}\right)$$

$$A_s \times \frac{m\,(1 - n)}{n} = \frac{Bt}{2}\left(\frac{(2\,nd - t)}{nd}\right) + \frac{b'}{2} \cdot \frac{(nd - t)^2}{nd} \quad . \; . \; (1)$$

Distance from neutral axis to centre of gravity of compressive stresses in the flange

$$= \frac{2\,nd + (nd - t)}{3\,(nd + (nd - t))} \times t + (nd - t)$$

$$= \frac{3\,nd - t}{3\,(2\,nd - t)} \times t + (nd - t).$$

Distance from neutral axis of centre of gravity of compressive stresses in the rib

$$= \frac{2}{3}\,(nd - t)$$

Taking moments about the centre of the tensile steel we have

applied moment = moment of resistance

$$M = \frac{Bt}{2} \times f_c\left(\frac{2\,nd - t}{nd}\right) \times \left(\frac{(3\,nd - t)}{3(2\,nd - t)} \times t + d - t\right)$$

$$+ \frac{b}{2} f_c \times \frac{(nd - t)^2}{nd} \times \left(\frac{2}{3}(nd - t) + (d - nd)\right) \quad . \; . \; (2)$$

These equations are not applicable to practical design. In the first place they are far too involved to be of any use and secondly there are special considerations in most cases of T-beams which render their use necessary.

Practical Application.—(1) The small amount of compression resisted by the compressed area of the rib may be ignored in comparison with that taken by the flange.

(2) The centre of pressure of the flange area is very nearly half way down the flange in all practical cases, i.e., $\frac{t}{2}$ distant from the compressed edge.

Thus the value of the lever arm of the resisting moments can be written down immediately with sufficient accuracy for practical design.

(3) As most T-beams occur in slab-and-girder floor construction where the actual floor-slab constitutes the top flange of the beams the value of B which may be taken as forming part of any particular beam depends on the position and direction of other beams in the floor.

It would be possible to treat the section on the lines of Chapter III with values of p_s and Bd^2 but such a method is neither necessary nor desirable. Owing to the way in which the sections occur in practice the best method to use is to treat f_c and f_s as fixed and treat B as variable.

Calculate the minimum value of B which will suffice to carry the compressive stresses. If this value is less than the allowable width that may be taken as forming part of the beam the design is sound. If however the minimum necessary value of B is larger than the allowable value then the design must be modified.

In designing a floor the slab is designed first. The value of t is therefore fixed before the design of the T-beams is started. The actual value chosen for d may be governed by considerations of headroom, size of other beams in floor, etc.

These considerations must be postponed as at this stage we are only concerned with the principles governing the design of sections.

Our method of design therefore is as follows :—

(*a*) The value of M is known.

(*b*) t is already fixed.

(*c*) Assume that f_c and f_s are both equal to the maximum allowable working stresses.

That is, if we are working to stresses of 750 and 18,000 lbs. per sq. in. assume that f_c equals exactly 750 and f_s equals exactly 18,000.

(*d*) The foregoing assumption at once fixes n.

(*e*) Assume a value for d (see Table TB II).

(*f*) Estimate the lever arm jd. This is very nearly equal to $\left(d - \frac{t}{2}\right)$

(*g*) Calculate the minimum value of B and check same.

(*h*) Calculate A_s.

Dimensions of T-Beams.—Some designers give the overall depth of the beam while others give the depth of the rib, i.e., the depth below the floor. Some quote the breadth of the rib first while others quote the depth first.

The beam shown in Fig. 20

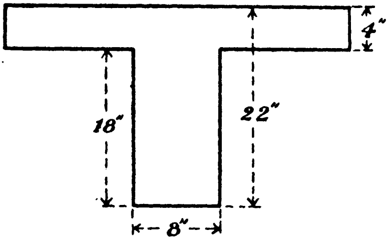


Fig. 20.

would be described as :—

8″ × 22″
8″ × 18″
22″ × 8″
18″ × 8″

according to the particular designer or firm.

The author has generally given the overall depth but has tried to make it plain in every case by writing for example 8″ × 22″ (overall).

Particular Case.—If t is fairly large compared with d it may happen that the neutral axis falls within the flange. Since the concrete below the neutral axis resists no stress the T-beam becomes equivalent to a rectangular beam as in Chapters II and III.

Failure by Tension in Steel.—It is theoretically possible to increase A_s indefinitely. Failure by tension is easily avoided by increasing A_s.

Failure by compression in the concrete.—If the minimum necessary value of B exceeds the allowable value the following courses are open

(1) Increase the value of t
(2) Increase the area of steel A_s
(3) Put in compression steel A_s'
(4) Increase d.

(1) Increasing the value of t increases the area under compression (provided that the neutral axis does not already fall inside the flange). Such an increase may, however, necessitate redesigning other parts of the structure.

(2) Increasing the area of tensile steel A_s lowers the value of f_s and causes the neutral axis to drop. This causes the allowable stress on the underside of the flange to increase, as the rate at which the stress in the concrete decreases becomes less (see Fig. 19). Thus the average stress on the concrete increases. This method is never used in practice.

(3) Introducing compression steel into T-sections is, as a rule, unsatisfactory. Tendency to failure by compression most often occurs in the main beams of floors. The bars in the floor-slab pass over the bars in the secondary beams which in turn pass over the bars of the main beams. Compression steel in main beams is not very effective therefore, as it cannot be placed very near the compressed edge. The large bars used in main beams have a tendency to buckle when in compression and must be tied down to the tensile steel and to one another by links otherwise they might burst out of the concrete. This necessitates grouping all the compression steel directly over the rib within the breadth b'. It cannot be spread out in the flange. The presence of large bars in such a position makes it difficult to concrete such a section and weakness may result through air pockets forming under the top bars.

If the difference between the minimum calculated value for B (without compression steel) and the allowable value is large the necessary area of compression steel to make up the deficiency becomes very large owing to the low working stress f_s'. Compression steel, therefore, should only be used as a last resort.

(4) Increasing d increases the lever arm, decreases the total compression and thereby decreases the value of B required. If considerations of size and headroom do not interfere this is the simplest solution.

Table TB I.—This table shows the safe total compression a flange will carry with stresses of 750 and 18,000 lbs. per sq. in. with $m = 18$. The preparation of such a table is simple.

With these stresses and $m = 18$

$$n = 0{\cdot}428$$

Thus if $d = 20''$ and $t = 4''$

$$nd = 8{\cdot}56 \text{ ins.}$$

The average compressive stress on the flange occurs in the middle, 2 ins. down from the top edge and by proportion is

$$\frac{6{\cdot}56}{8{\cdot}56} \times 750 = 575 \text{ lbs. per sq. in.}$$

If $B = 1$ in. the total area of the flange is 1 in. × 4 ins. or 4 sq. ins.

The total compression is 4×575

$= \mathit{2300\ lbs.}$ for $B = 1$ ins.

giving a point on Table TB I.

Practical Sizes.—It is possible for special cases to use very broad shallow beams or deep narrow ones; to use very small or very large percentages of steel. For ordinary floor work the rib of the beam which projects below the floor is generally about twice as deep as it is wide. Thus net dimensions (see Fig. 20) of 7 ins. × 14 ins., 8 ins. × 16 ins., 9 ins. × 18 ins., 10 ins. × 20 ins., and 14 ins. × 24 ins. are very common. The smaller sections used for secondary beams have usually 2% to 3% of tensile steel reckoned on the $b'd$ area (Fig. 19) and the larger sections used for main beams 4% to 5%. The beginner usually expects the bars in the section to be arranged as in Fig. 21, but from practical considerations they are better arranged as Fig. 22, Fig. 23 or Fig. 24. The use of deep narrow beams with two large bars as Fig. 25 was once fairly common, but is now, for practical reasons, seldom seen.

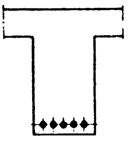


Fig. 21.

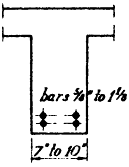


Fig. 22.

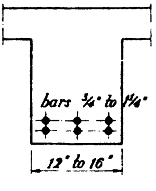


Fig. 23.

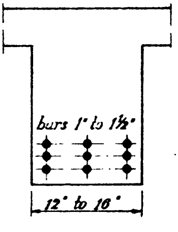


Fig. 24.

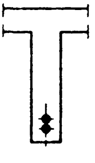


Fig. 25.

Although it is, perhaps, more logical to fix d first then calculate A_s, it is often neater from a practical standpoint to choose a neat practical number and size of bars and then calculate d to suit.

If the reader finds difficulty in choosing a section he may refer to Table TB II, aiming at the middle value in each horizontal line and arranging the bars as in Figs. 22, 23 or 24.

Examples.—A T-section has a flange 4 ins. thick, and a maximum allowable width B of 40 ins. It has to resist a moment of 600,000 lbs.-ins.

If f_c is not to exceed 750 and f_s not to exceed 18,000 lbs. per sq. in. with $m = 18$, find a suitable section.

Looking at Table TB II a section with a lever arm of 14 ins. seems indicated. We may therefore try a section as in Fig. 26. With the values of f_c, f_s and m we have $n = 0{\cdot}428$

$$nd = 0{\cdot}428 \times 16''$$

$$= 6{\cdot}85''$$

$$jd = \text{say } 14''$$

$$\text{Total compression} = \text{Total tension}$$

$$= \frac{600{,}000}{14} = 42{,}900 \text{ lbs.}$$

$$\text{Average compression on flange} = \frac{4{\cdot}85}{6{\cdot}85} \times 750$$

$$= 530 \text{ lbs. per sq. in.}$$

Total safe compression per 1 in. width of B

$$= 530 \times 4 = 2120 \text{ lbs.}$$

(This might be read directly from Table TB I.)

$$\text{Min}^{\text{m}}\text{. width of } B \text{ required} = \frac{42{,}900}{2120} = 20{\cdot}3 \text{ ins. only}$$

$$A_s = \frac{42{,}900}{18{,}000} = 2{\cdot}37 \text{ sq. ins.}$$

Say 4 bars ⅞" (2·40 sq. ins.).

A T-section has a flange 5 ins. thick with a maximum allowable B of 60 ins. It has to resist a moment of 1,000,000 lbs.-ins. with stresses not exceeding 900 and 16,000 lbs. per sq. in. with $m = 14$. Find a section.

Referring to Table TB II and remembering that this table is made for $f_s = 18{,}000$, we may try a section as in Fig. 27.

With the given stresses of 900 and 16,000

$$n = 0{\cdot}442$$

$$nd = 0{\cdot}442 \times 18'' = 7{\cdot}95 \text{ ins.}$$

$$jd = 15{\cdot}75'' \text{ say}$$

$$\text{Total compression} = \text{Total tension}$$

$$= \frac{1{,}000{,}000}{15{\cdot}75} = 63{,}500 \text{ lbs.}$$

$$\text{Average compression on flange} = \frac{5{\cdot}45}{7{\cdot}95} \times 900 = 615 \text{ lbs. per sq. in.}$$

Strength of 5-in. flange = 3045 lbs. per 1 in. of B

$$\text{Minimum } B \text{ required} = \frac{63{,}500}{3045} = 21 \text{ ins. only.}$$

$$A_s = \frac{63{,}500}{16{,}000} = 3{\cdot}97 \text{ sq. ins. say 4 bars } 1\tfrac{1}{8} \text{ ins.}$$

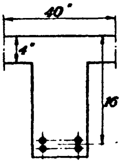


Fig. 26.

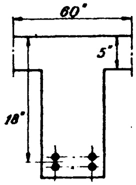


Fig. 27.

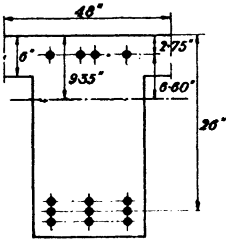


Fig. 28.

A T-section has a 6-in. flange with a maximum B of 48 ins. The overall depth of the section is not to exceed 30 ins. and it has to resist a moment of 3,300,000 lbs.-ins. with stresses of 600 and 16,000 and $m = 15$. Any compression steel required must be placed 2¾ ins. down from the top edge. What reinforcement is required ?

Using the arrangement in Fig. 28

$$d = 26 \text{ ins.}$$

$$jd = 23\tfrac{1}{2} \text{ ins. say.}$$

With the given stresses $n = 0{\cdot}36$.

$$nd = 0{\cdot}36 \times 26'' = 9{\cdot}35 \text{ ins.}$$

$$\text{Total compression} = \frac{3{,}300{,}000}{23{\cdot}5} = 141{,}000 \text{ lbs.}$$

Average compressive stress on flange

$$= \frac{6{\cdot}35''}{9{\cdot}35''} \times 600 = 406 \text{ lbs. per sq. in.}$$

Compressive strength of 6-in. flange = 2436 lbs. per 1 in. of B.

Total compressive strength of flange if B is 48 ins.

$$= 48 \times 2436 = 116{,}000 \text{ lbs.}$$

Total compression to be taken by compression steel

$$= 141{,}000 - 116{,}000$$

$$= 25{,}000 \text{ lbs.}$$

$$\text{Effective } f_s' = (15 - 1) \times \frac{6{\cdot}60''}{9{\cdot}35''} \times 600 = 5910 \text{ lbs. per sq. in.}$$

$$\text{Area } A_s' \text{ required} = \frac{25{,}000}{5910} = 4{\cdot}2 \text{ sq. ins.}$$

say 4 bars 1⅛ ins.

$$A_s = \frac{141{,}000}{16{,}000} = 8{\cdot}80 \text{ sq. ins.}$$

Say 9 bars 1⅛ ins.

Tables TB I and TB II.—It is doubtful if such tables save any time in the drawing office as the values they show may be so readily calculated on the slide-rule. The preparation of such tables, on the other hand, will give the reader valuable familiarity with the subject and he may draw up Table TB I for $f_c = 950$, $f_s = 18{,}000$ and $m = 14$ and Table TB II for $f_s = 16{,}000$ or any other values of f_c, f_s and m to which he may want to work.

⅂-Sections.—The bressummers round the edge of a floor are usually of this section. As the section is not symmetrical a bending moment applied in a vertical plane produces torsion. This is resisted by the floor slab and any beams which frame into the bressummer. In general their design is similar to that employed for T-sections with the addition that the torsion must be taken into account. Further discussion must be postponed until the Design of Members as the stresses involve other parts of the structure.

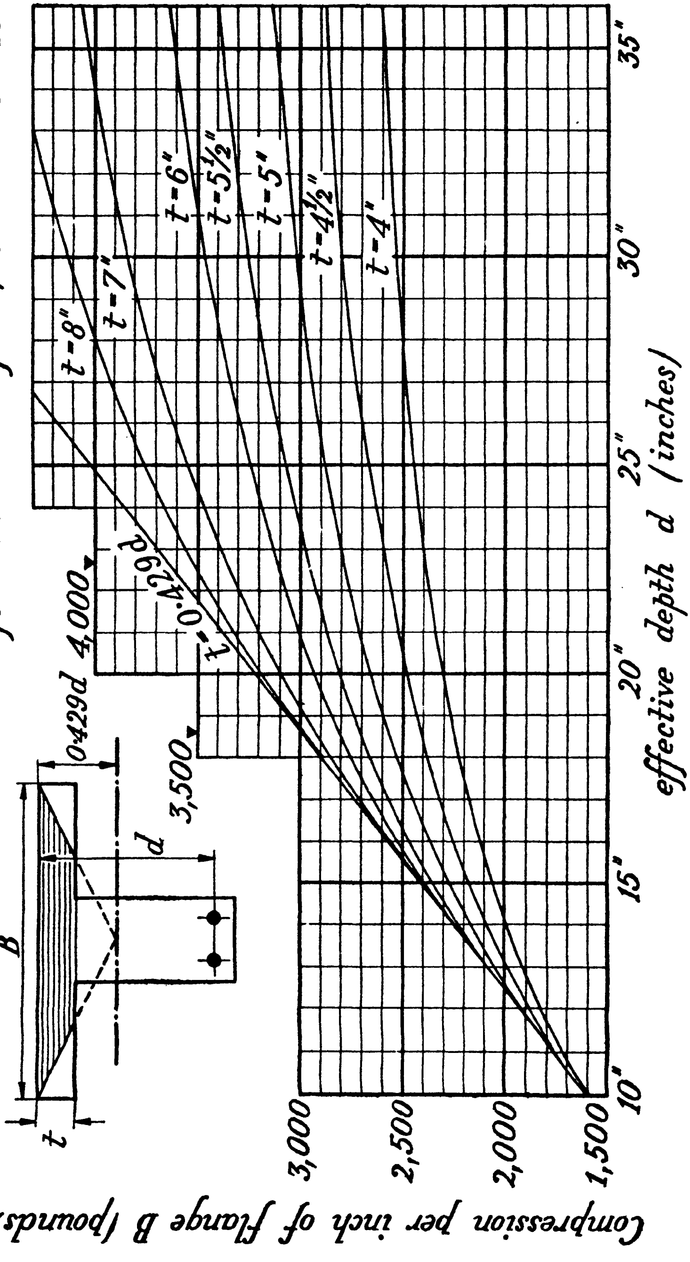


TABLE TB I

Value of compression flange of T-beam
fc = 750 fs = 18,000 m = 18

TABLE TB II.

MOMENTS OF RESISTANCE OF GROUPS OF BARS WITH LEVER ARM jd IF $f_s = 18{,}000$ LBS. PER SQ. IN.

	$jd = 10''$	$jd = 12''$	$jd = 14''$	$jd = 16''$	$jd = 18''$	$jd = 20''$	$jd = 22''$	$jd = 24''$	$jd = 26''$	$jd = 28''$
4 Bars $\frac{5}{8}''$	220,000	265,000	308,000	350,000						
4 Bars $\frac{3}{4}''$	318,000	382,000	446,000	510,000	573,000					
4 Bars $\frac{7}{8}''$	434,000	520,000	607,000	695,000	782,000	865,000				
4 Bars 1″	565,000	680,000	792,000	905,000	1,020,000	1,130,000	1,240,000			
4 Bars $1\frac{1}{8}''$		862,000	1,010,000	1,150,000	1,290,000	1,440,000	1,580,000	1,730,000		
6 Bars $\frac{7}{8}''$		780,000	910,000	1,040,000	1,170,000	1,300,000	1,430,000	1,560,000		
6 Bars 1″			1,190,000	1,360,000	1,530,000	1,700,000	1,870,000	2,040,000	2,210,000	
6 Bars $1\frac{1}{8}''$			1,500,000	1,710,000	1,930,000	2,150,000	2,360,000	2,570,000	2,790,000	
6 Bars $1\frac{1}{4}''$			1,860,000	2,120,000	2,390,000	2,650,000	2,920,500	3,180,000	3,450,000	3,710,000
8 Bars 1″			1,580,000	1,810,000	2,040,000	2,260,000	2,490,000	2,710,000	2,940,000	3,160,000
8 Bars $1\frac{1}{8}''$				2,290,000	2,570,000	2,860,000	3,150,000	3,430,000	3,720,000	4,000,000
8 Bars $1\frac{1}{4}''$				2,830,000	3,190,000	3,530,000	3,890,000	4,250,000	4,620,000	4,950,000
9 Bars 1″				2,030,000	2,300,000	2,550,000	2,810,000	3,060,000	3,320,000	3,570,000
9 Bars $1\frac{1}{8}''$				2,570,000	2,900,000	3,220,000	3,540,000	3,860,000	4,180,000	4,500,000
9 Bars $1\frac{1}{4}''$					3,580,000	3,960,000	4,370,000	4,760,000	5,170,000	5,560,000
9 Bars $1\frac{3}{8}''$					4,050,000	4,800,000	5,300,000	5,770,000	6,250,000	6,720,000
9 Bars $1\frac{1}{2}''$					5,150,000	5,730,000	6,300,000	6,890,000	7,450,000	8,020,000

All Moments of Resistance are in Pounds-Inches.

CHAPTER V

SECTIONS UNDER TENSION ONLY

Where tension members are in a protected position such as ties to sheet piling buried in the ground, their section may be as shown in Fig. 29, with one large bar in the centre. If, however, the member may receive accidental loads due to such causes as collisions with run-away skips, blows from falling merchandise, etc., the section should be as shown in Fig. 30 as this gives some moment of resistance.

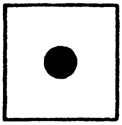


Fig. 29.

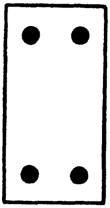


Fig. 30.

In either case we have, as the concrete can resist no tension,

$$\text{Area of steel} = \frac{\text{Total tension}}{f_s}$$

where f_s is 16,000 to 18,000 lbs. per sq. in.

Watertightness.—If the section in question is part of a watertight wall such as frequently occurs in circular tanks, a calculated stress of 16,000 to 18,000 lbs. per sq. in. allows excessive widening of shrinkage cracks or construction joints (intentional or accidental) and leakage. In this case a satisfactory result is obtained by writing

$$\text{Area of steel} = \frac{\text{Total tension}}{f_s}$$

where f_s is 12,000 to 13,500 lbs. per sq. in. In cases of isolated tension members the placing of the steel in the section is often decided by questions of bar spacing, grip-length and the design of the joints where the member runs into the rest of the structure.

These considerations will be gone into later.

CHAPTER VI

SECTIONS UNDER COMPRESSION ONLY

Alternative Methods of Design.—Up till recent years it has been usual to design sections under pure compression in accordance with the assumptions outlined in Chapter I. It is now suggested that their design should be based on the conditions which exist when the section reaches its ultimate strength. The author has always upheld the principle that the *ultimate* strength is the only real criterion of the carrying capacity of any structure. On the other hand while all other sections are designed on working stresses, it is very awkward to design compression sections alone on ultimate stresses.

Working Stress Method—Analysis.—Let Fig. 31 represent a section carrying a pressure L at its centroid.

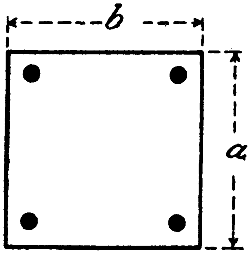


Fig. 31.

It will be seen that such a section should be symmetrically reinforced in order to avoid accidental bending stresses.

Let A = area of steel

p = percentage area of steel $= \frac{A}{ab} \times 100$

f_c = stress on concrete

f_s = stress on steel.

Since the load is applied at the centroid the value of f_c is uniform over the whole section.

$\therefore$ Total compression resisted by concrete

$$= \text{Area} \times \text{stress}$$
$$= (ab - A)\,f_c$$

Total compression resisted by steel

$$= Af_s \text{ and since } f_s = mf_c$$

this $= mAf_c$

Total compression resisted by the whole section

$$= (ab - A)\,f_c + mAf_c$$
$$= \text{Total applied load}$$
$$= L$$
$$\therefore L = (ab - A)\,f_c + mAf_c$$
$$= f_c\,(ab + (m - 1)\,A)$$
$$= f_c\,ab\left[1 + \frac{(m-1)}{100}\,p\right].$$

Thus the average stress on the whole section

$$= f_c\left[1 + \frac{(m-1)}{100}\,p\right].$$

This result is independent of the shape of the section and applies equally well to a circular section such as Fig. 32.

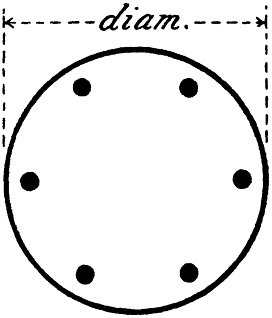


Fig. 32.

Stress on Steel.—If $f_c = 600$ lbs. per sq. in. then $f_s = m \times 600$. If $m = 15$

$$f_s = 9000 \text{ lbs. per sq. in.}$$

There is thus no danger of the steel being overstressed, the carrying power of the section being determined entirely by the value of f_c.

Ultimate Stress Method.—It has long been known (see Chapter XVI) that the ultimate strength of a section such as Fig. 31 or Fig. 32 under a central load is equal to the area of concrete multiplied by the unit crushing strength of a plain concrete cylinder, plus the area of steel multiplied by its elastic limit in compression. This latter value is not easy to define but for mild steel is about 40,000 lbs. per sq. in. The unit crushing strength of a plain concrete cylinder whose height exceeds twice its diameter is about 80% of the unit crushing strength of a cube made from the same concrete. If we have a 1 : 2 : 4 concrete, having a test-cube strength of 2250 lbs. per sq. in. a cylinder of plain concrete would have a crushing strength of 0·80 × 2250 or 1800 lbs. per sq. in. The ultimate strength of a section such as Fig. 31 or Fig. 32 made from this concrete would be

Ultimate $L = 1800 \times$ (area of concrete) $+ 40{,}000$ (area of steel). Using a factor of safety of 3 :—

Safe working $L = 600 \times$ (area of concrete) $+ 13{,}500$ (area of steel) or

Safe $L = 600 \times$ (gross area) $+ 12{,}900$ (area of steel).

Hooping.—All columns in addition to longitudinal bars, which constitute the main reinforcement, have also a secondary reinforcement consisting of links or hoops.

Fig. 33 shows both types.

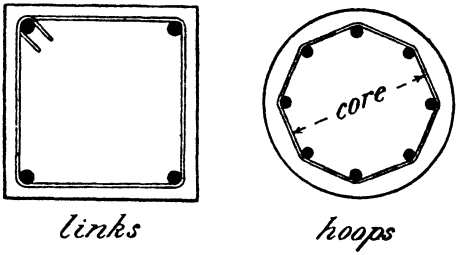


Fig. 33.

The object of providing these is twofold.

(1) To prevent the main bars from buckling and bursting out of the concrete.

(2) to prevent the concrete shearing and cracking on planes inclined at 45° to the axis in the same way as a plain concrete cube fails under compression in a testing machine. In other words to make the column " tough " instead of brittle.

Main Reinforcement (Longitudinal).—It can be shown that steel stressed to 9000 lbs. per sq. in. is dearer than concrete stressed to 600 lbs. per sq. in. For example, one square inch of steel weighs about 4 lbs. per ft. run in a column including laps. With steel at £16 per ton in place, one square inch area will cost about 4 × 1·7 = 6·8 pence per foot. One square inch of concrete with concrete at 60s. per cub. yd. costs $\frac{12}{1728} \times \frac{1}{27} \times 60 \times 12 = \frac{5}{27}$ pence per foot.

15 square inches cost 2·8 pence per foot run as against 6·8 pence for one square inch of steel. Thus concrete is less than half as costly as steel to carry the same load. This aspect of the question has, as a rule, little bearing on practical column design.

The main points to look after are :—

(1) The bars must be stiff enough to stand up straight in the column boxes during concreting.

(2) They must be stiff enough to take the compression stress without buckling outwards, and must at the same time help to prevent the concrete splitting outwards along its planes of greatest shear.

(3) They must be sufficiently small and few in number to allow proper concreting of the member and reasonably simple joints at its ends.

Consideration No. (1) is easily achieved by using bars of ½-in. diameter and upwards.

Consideration No. (3) is assured by limiting the size of bars to 1½ in. diameter as a maximum and limiting the percentage area of steel to 4% or 5% according to circumstances.

Consideration No. (2) is intimately associated with the number and spacing of links or hoops which is discussed below.

Secondary Reinforcement (Links or Hoops).—It has been put forward that a circular column with a large number of closely spaced hoops or spirals as in Fig. 34 has a much larger ultimate strength than a column having only a few small links as the section in Fig. 34.

The facts of the matter appear to be as follows.

When a hooped section column is tested *in a testing machine* it behaves exactly as a rodded section until the concrete reaches the ultimate stress that plain concrete will carry (about 2400 lbs. per sq. in. for a 1 : 2 : 4 mixture).

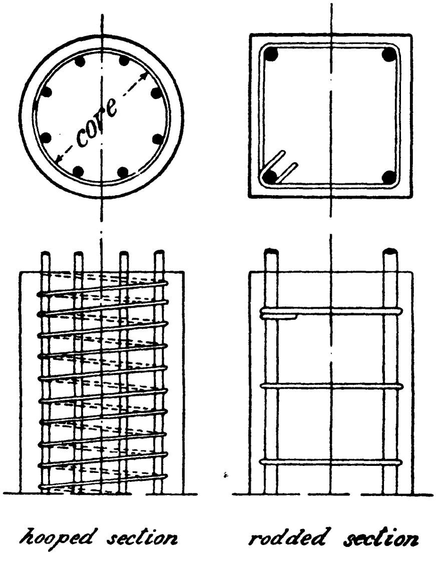


Fig. 34.

At this stage the rodded column fails and disintegrates.

The hooped section begins at this stage to grow rapidly shorter and the concrete outside the hoops cracks and comes away. If more load is applied the column continues to shorten and may eventually " fail " by buckling of the whole section or bursting of the hoops. The stress per square inch that can be applied before ultimate failure in a testing machine varies with the amount of hooping provided.

It has been claimed :—

(1) That the spalling off of the concrete outside the hoops is a useful " danger signal " prior to failure.

(2) That the behaviour of a hooped section under load justifies a higher working stress in such a column.

Both these statements appear to the author to be fallacious.

A designer should be absolutely certain of the strength of his designs and the provision of any " danger signal " is unnecessary.

No engineer would allow a column to remain in a structure after the outer layer of concrete had spalled off. Moreover, the excessive shortening shown by hooped sections in a testing machine would inevitably cause total failure of the beams framing into such a column in an actual structure. It appears therefore to the author that a hooped column which is beginning to spall and shorten under load has, for all practical purposes, already " failed".

Experiments carried out in America seem to show that the concrete in a hooped section is in a state of flux after the spalling point is reached and that the column will actually fail under the load at this point, provided it is kept continually stressed, although final failure may be delayed twenty or thirty hours. The extra load which such a section will carry in a testing machine is therefore due to the rapidity of loading and constitutes a laboratory phenomenon which is without practical significance. Links equal in volume to 0·20% of the concrete volume are ample to guard against any brittleness of the column under static load.

The effectiveness of close hooping or links against sudden loads is well illustrated in standard methods of reinforcing the heads of piles (see Chapter XXI).

Standard Practice.—In cases where the size of the column is not restricted, a square column having about 1% area of main steel is employed. For small columns four bars, one in each corner, will suffice. For larger sizes eight or twelve bars may be used. Round, hexagonal, etc., columns occur less frequently at the present time but the introduction of steel centering may lead to extended use of circular sections.

In cases where the size of the column must be kept down, the area of main steel may be as high as 5%.

The links should not be farther apart than about 16 diameters of the main bars and not farther apart than about half the least diameter of the column.

The minimum amount of links to be provided is quite clear from standard practice. Millions of columns, including pre-cast piles have been constructed with main bars of 0·75% to 2% of the cross-section and links equal in volume to 0·20% of the total volume of concrete or less. The author does not know of a single case where failure has developed due to insufficient volume of links during loading or during pile driving, and it is quite clear that 0·20% of links is quite sufficient if the main bars do not exceed 2%. Very few columns have more than 2% of main steel and the minimum volume of links required when the main bars are 3%, 4% or 5% is not clear, but provision of 0·30% for 3% main steel and 0·4% for 4% or 5% of main steel will be on the safe side.

The author is, of course, aware that some Regulations call for 0·4% to 0·5% of links for 1% of main bars, but there is no practical reason for such extravagant proposals.

Fig. 35 shows some usual sections.

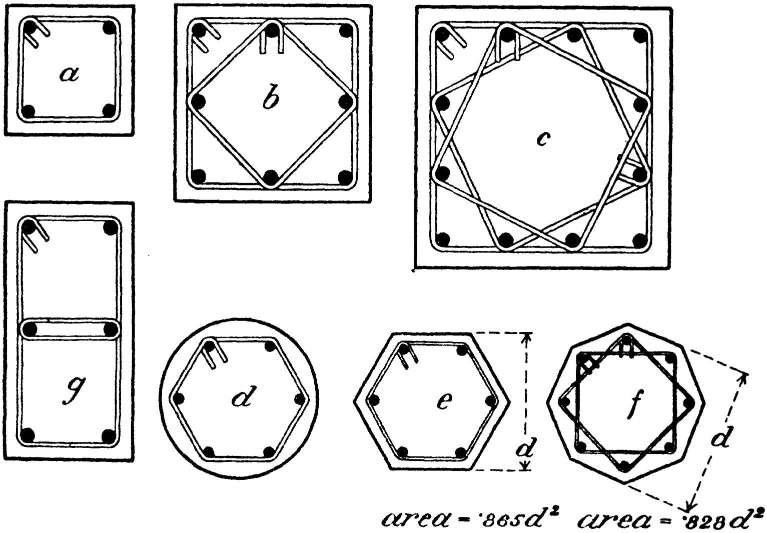


Fig. 35.

Volume and Weight of Links.—In reckoning the volume of links the short end hooks should not be included. For example, suppose we have a column section as in Fig. 36 measuring 18 ins. × 18 ins., and suppose the links are ⅜-in. diam. and have four sides each 1 ft. 3¼ ins. long and two end hooks 2½ ins. long (nominal). The overall length of link is 4 × (1 ft. 3¼ ins.) + 2 × 2½ ins. or 5 ft. 6 ins. and its weight is 5·5 × 0·376 = 2·07 lbs., but the effective length for purposes of calculating the volume is 4 × 1 ft. 3¼ ins. = 5 ft. 1 in. or 61 ins. and the effective volume is 61 × 0·11 = 6·7 cu. ins. If the links were spaced at 6-in. centres there would be one link with a volume of 6·7 cu. ins. to a volume of concrete 18 ins. × 18 ins. × 6 ins. high, that is 6·7 cu. ins. of steel to 1950 cu. ins. of concrete.

The percentage volume of link is :—

$$\frac{6{\cdot}7 \times 100}{1950} = 0{\cdot}345\%.$$

The weight of links per yard run of column would be six links of 2·07 lbs. each or 12·4 lbs. per yard. To save calculation in the standard cases in Figs. 36 to 38, the percentages and weights of links are very closely as follows.

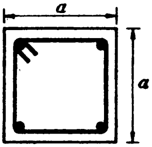


Fig. 36.

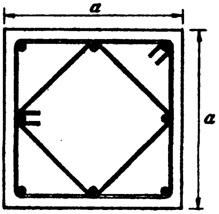


Fig. 37.

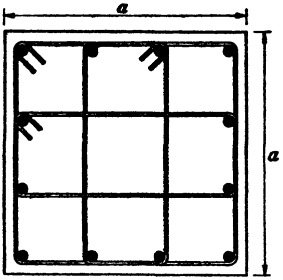


Fig. 38.

As Fig. 36

For $\frac{3}{16}$-in. diam. links :—

$$\text{percentage} = \frac{(11\,a - 30)}{a^2 \times (\text{spacing})} \quad \text{weight per yard} = \frac{(1{\cdot}12\,a - 1{\cdot}7)}{\text{spacing}}\ \text{lbs.}$$

For $\frac{1}{4}$-in. diam. links :—

$$\text{percentage} = \frac{(19{\cdot}6\,a - 54)}{a^2 \times (\text{spacing})} \quad \text{weight per yard} = \frac{(2\,a - 3)}{\text{spacing}}\ \text{lbs.}$$

For $\frac{3}{8}$-in. diam. links :—

$$\text{percentage} = \frac{(44\,a - 121)}{a^2 \times (\text{spacing})} \quad \text{weight per yard} = \frac{(4{\cdot}5\,a - 6{\cdot}75)}{\text{spacing.}}\ \text{lbs.}$$

As Fig. 37

For $\frac{1}{4}$-in. diam. links :—

$$\text{percentage} = \frac{(34\,a - 92)}{a^2 \times \text{spacing}} \quad \text{weight per yard} = \frac{(3{\cdot}43\,a - 4{\cdot}4)}{\text{spacing}}\ \text{lbs.}$$

For $\frac{3}{8}$-in. diam. links :—

$$\text{percentage} = \frac{(76\,a - 208)}{a^2 \times (\text{spacing})} \quad \text{weight per yard} = \frac{(7{\cdot}7\,a - 10)}{\text{spacing}}\ \text{lbs.}$$

As Fig. 38

For $\frac{1}{4}$-in. diam. links :—

$$\text{percentage} = \frac{(46\,a - 108)}{a^2 \times (\text{spacing})} \quad \text{weight per yard} = \frac{(4{\cdot}67\,a - 3{\cdot}5)}{\text{spacing}}\ \text{lbs.}$$

For ⅜-in. diam. links :—

$$\text{percentage} = \frac{(103\,a - 240)}{a^2 \times (\text{spacing})} \quad \text{weight per yard} = \frac{(10{\cdot}5\,a - 8)}{\text{spacing}}\ \text{lbs.}$$

In all the above the dimensions of the column and the spacing must be expressed in inches.

EXAMPLES

A column is to carry a central load of 100 tons with a stress not exceeding 900 lbs. per sq. in. on the concrete and 13,500 lbs. per sq. in. on the steel. If the size is unrestricted find a suitable square section with 0·20% links.

Assuming a square section as Fig. 36 with 1% of steel

Concrete area $= 0{\cdot}99\,a^2$

Steel area $= 0{\cdot}01\,a^2$

$$\begin{aligned} \text{Safe load} &= (900 \times 0{\cdot}99\,a^2 + 13{,}500 \times 0{\cdot}01\,a^2)\ \text{lbs.} \\ &= 100\ \text{tons} \\ \therefore a &= 14{\cdot}75\ \text{ins.} \end{aligned}$$

and area of steel $= 2{\cdot}18$ sq. ins.

Say a 15 in. × 15 in. section with four bars ⅞-in. (2·4 sq. ins.) or four bars ¾ in. (1·76 sq. ins.) in competitive design.

Using ¼-in. diam. links :—

$$\text{percentage} = \frac{(19{\cdot}6\,a - 54)}{a^2 \times \text{spacing}}$$

$$0{\cdot}20 = \frac{(19{\cdot}6 \times 15) - 54.}{15^2 \times \text{spacing}}$$

$$\text{spacing} = 5{\cdot}32\ \text{ins.} \quad \text{say 5 ins. spacing.}$$

A column is to carry a central load of 600,000 lbs. The safe stress on the concrete is 800 lbs. per sq. in. and the safe stress on the steel is 13,500 lbs. per sq. in. To save space the column is to have about 5% of main steel. Design a suitable square section with 0·5% links.

Assuming a square section as Fig. 38

$$\text{Concrete area} = 0{\cdot}95\,a^2$$

$$\text{Steel area} = 0{\cdot}05\,a^2$$

$$\begin{aligned} \text{Safe load} &= (800 \times 0{\cdot}95\,a^2 + 13{,}500 \times 0{\cdot}05\,a^2)\ \text{lbs.} \\ &= 600{,}000\ \text{lbs.} \\ \therefore a &= 20{\cdot}5\ \text{ins.} \end{aligned}$$

Using a 20 in. × 20 in. column, the gross area is 400 sq. ins. If this were all concrete it would carry 320,000 lbs.

Adding one square inch of steel adds (13,500 — 800) lbs. to the net strength or 12,700 lbs.

$$\text{Area of steel required} = \frac{600{,}000 - 320{,}000}{12{,}700} = 22 \text{ sq. ins.}$$

Using a 21 ins. × 21 ins. column

$$\text{Area of steel required} = \frac{600{,}000 - 353{,}000}{12{,}700} = 19{\cdot}5 \text{ sq. ins.}$$

Say a column 20 ins. × 20 ins. with 12 bars 1½ in. diam. (21·2 sq. ins.) or a column 21 in. × 21 ins. with 8 bars 1½ ins. diam. and 4 bars 1¼ in. diam. (19·01 sq. ins.).

Using ⅜-in. links to give 0·5% volume.

$$\text{For a 20 ins.} \times \text{20 ins. column} \quad 0{\cdot}5 = \frac{(103 \times 20) - 240}{20^2 \times \text{spacing}}$$

giving a spacing of 9·1 ins. say 9 ins.

$$\text{For a 21 ins.} \times \text{21 ins. column} \quad 0{\cdot}5 = \frac{(103 \times 21) - 240}{21^2 \times \text{spacing}}$$

giving a spacing of 8·65 ins. say 8½ ins.

CHAPTER VII

SECTIONS UNDER BENDING AND TENSION COMBINED

We have seen in previous chapters that a bending moment alone always produces compressive stresses at one edge of the section and tensile stresses at the other. On the other hand a tensile stress alone is produced in a section subject only to a tensional pull.

In sections subject to a combination of the two the nature of the stresses set up will depend on the relative proportions of bending and direct stress. If the direct pull is large and the bending moment small then the stress will be tensional everywhere although this stress will vary in intensity. If the bending moment is relatively large then compressive stresses will occur over part of the section. These two cases will be treated separately.

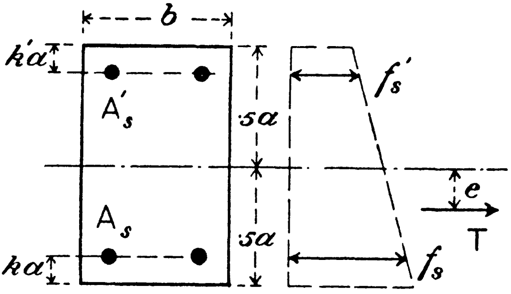


Fig. 39.

Case I. No compressive stress on section.—Let Fig. 39 represent a section under a tensile pull T applied at a distance *e* from the centre of the concrete section.

The section is reinforced with areas :—

A_s of steel near the more heavily stressed edge.

A_s' of steel near the less heavily stressed edge.

(The same notation has been employed as for beams although in this case f_s' is a tensile stress.)

Since the stress is everywhere tensional the concrete takes no part in resisting it

Total stress on top steel $= A_s' f_s'$

Total stress on bottom steel $= A_s f_s$.

The sum of these two must be equal to the total tension on the section.

$$\text{i.e.}\quad A_s'f_s' + A_sf_s = T \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Taking moments about the centre of the top steel.

Moment of applied forces = Moment of resisting forces

$$T \times (e + a/_2 - k'a) = A_sf_s(a - ka - k'a) \quad . \quad . \quad . \quad . \quad (2)$$

Similarly by taking moments about the centre of the bottom steel we have

$$T \times (a/_2 - e - ka) = A_s'f_s' \times (a - ka - k'a) \quad . \quad . \quad (3)$$

It appears therefore that if we know the size of the section and the location of the steel in the section we can calculate the values of A_s and A_s' necessary to make $f_s = f_s' = 16{,}000$ or any other desired value.

Example. if

$$a = 12 \text{ ins.}$$
$$ka = k'a = 1{\cdot}5 \text{ ins.}$$
$$T = 36{,}000 \text{ lbs.}$$
$$e = 3 \text{ ins.}$$
$$f_s = f_s' = 16{,}000$$

We have at once by moments about the top steel

$$T \times (3'' + 6'' - 1{\cdot}5'') = 16{,}000\, A_s\, (12'' - 1{\cdot}5'' - 1{\cdot}5'')$$

$$\text{i.e.}\quad A_s = \frac{36{,}000}{16{,}000} \times \frac{7{\cdot}5''}{9''} = \mathit{1{\cdot}87\ sq.\ ins.}$$

and by moments about the bottom steel

$$A_s' = \frac{36{,}000}{16{,}000} \times \frac{1{\cdot}5''}{9''} = \mathit{{\cdot}375\ sq.\ ins.}$$

In practice it often happens that the applied stresses are first calculated as a bending moment and a central tension. This case is easily resolved into an eccentric tension by writing

$$e = \frac{M}{T} \quad \text{and} \quad M \text{ then} = 0$$

as shown in Figure 40 which is a well-known theorem in elementary applied mathematics.

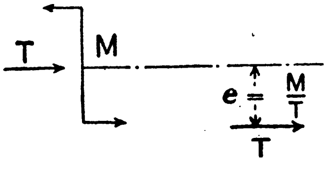


Fig. 40.

Limit of Case I.—It will be seen that when the stress is zero at the top edge then by similar triangles the relative values of f_s' and f_s are as their distances from this edge

$$\text{i.e. } \frac{f_s'}{f_s} = \frac{k'a}{a - ka} \text{ as in Fig. 39.}$$

If $A_s' = 0$ and there is no compressive stress on the concrete, then the tension T must act along the centre line of the tensile steel A_s.

Application.—The determination of a and b in Fig. 39 must be left until later as they will depend on the unsupported length of the member and its end connections.

It is obvious that the ratio of A_s to A_s' may be increased or diminished in any ratio by decreasing or increasing the value of a. The most economical values can only be determined by a consideration of all the facts.

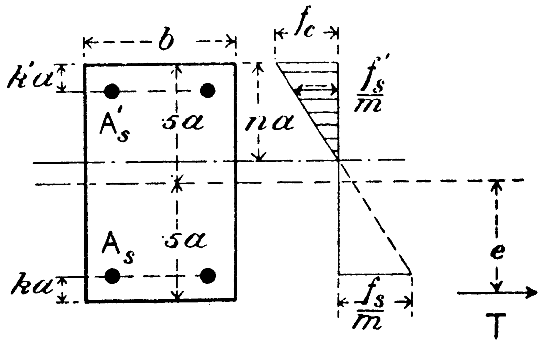


Fig. 41.

Case II. Compressive stresses occur.—Let Fig. 41 represent a section under a tensile pull T applied at a distance e from the centre of the concrete section.

Let the neutral axis occur at a depth $= na$ (this does not correspond exactly to the treatment of beams where n is a fraction of the effective, not overall, depth). By similar triangles

we have
$$\frac{f_s'}{m} = f_c \times \frac{(na - k'a)}{na}$$

$$\frac{f_s}{m} = \frac{f_c \times (a - ka - na)}{na}.$$

Total compression on stressed area of concrete

$$= b \times na \times \frac{f_c}{2}.$$

Total stress on $A_s' = A_s' f_s'$

$$= A_s' \times \frac{(na - k'a)}{na} \times mf_c.$$

Total stress on A_s

$$= A_s \times \frac{(a - ka - na)}{na} \times mf_c.$$

Equating the applied tension to the resisting forces :—

$$T = A_s \frac{(a - ka - na)}{na} \times mf_c - A_s' \times \frac{(na - k'a)}{na} (m - 1) f_c - \frac{bnaf_c}{2}.$$

Also by taking moments about the centre-line of the tensile steel we have :—

$$T \times \left(e - \left(\frac{a}{2} - ka\right)\right)$$

$$= \frac{bnaf_c}{2} \times \left(a - ka - \frac{na}{3}\right) + A_s' \times \frac{(na - k'a)}{na} (m - 1) f_c \times (a - ka - k'a).$$

We thus have two main equations. If the section is known beforehand we must solve for the two unknowns f_c and n. This is theoretically possible but quite out of the question in practice. The analysis however serves to indicate the lines on which the problem can be attacked.

Owing to the way in which the problem arises in practice it is best to subdivide it for treatment into two cases.

Case II (a).—Where e is fairly small and a symmetrical section is essential as in the case of a member which is essentially a tie but which may have a small transverse load in either direction. This needs a table such as Table T I.

Case II (b).—Where e is large or where a symmetrical section is not required as in the case of a tie with a transverse load which acts always in one direction. This case may be solved as follows. Assume a trial section as shown in Fig. 42. Transfer the tension T to the line of the tensile steel A_s and modify M to $(M - Th)$ accordingly. Supply sufficient A_s' and A_s to reinforce the section against the moment $(M - Th)$ alone and calculate f_s. These values are given in Tables RB I to RB VII. Add sufficient extra A_s to take T alone at this stress f_s which we have calculated. It is essential that the A_s added to resist the moment $(M - Th)$ must work at the same stress as the A_s added to resist T as all the bars are the same distance from the neutral axis.

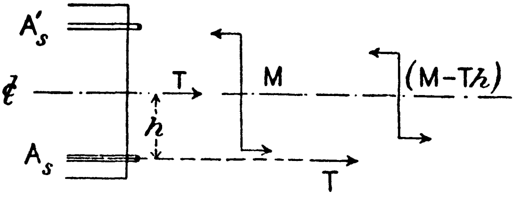


Fig. 42.

As in the case of rectangular sections under bending, any number of different sections can be designed with varying values of A_s' and A_s to resist combined stresses. The questions which limit practical choice of sections are discussed later.

Case II (a), Table T I.—To reduce the number of unknowns in our main equations we may assume $k = k' = 0{\cdot}1$ (see Fig. 41). The values are worked backwards by first assuming a value for n and a value for p and finding what values of T and e correspond to these assumptions. Table T I has been calculated for $m = 15$, but very little error is incurred by using it for $m = 18$ or $m = 12$.

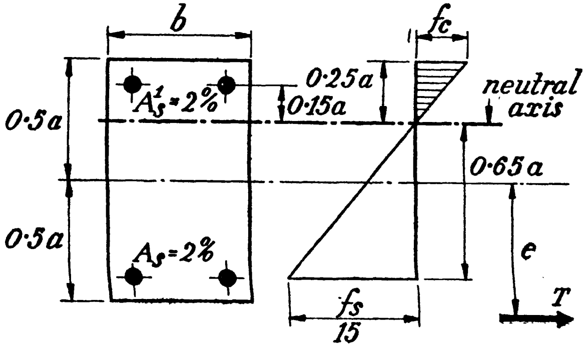


Fig. 43

Assume $n = 0{\cdot}25$ (as in Fig. 43)
and assume $A_s' = A_s = 2\%$ of ab or $p = 4\%$.
with $m = 15$

Following Fig. 41 we have

$$f_s' = \frac{0{\cdot}15}{0{\cdot}65} \times f_s$$

Effective f_s' (since one square inch of A_s' displaces one square inch of concrete) $= \frac{14}{15} \times \frac{0{\cdot}15}{0{\cdot}65} f_s = 0{\cdot}215 f_s$

$$f_c = \frac{0{\cdot}25}{0{\cdot}65} \times \frac{1}{15} \cdot f_s = 0{\cdot}0256 f_s.$$

The centre of pressure on the concrete is one-third of $0{\cdot}25\,a$
or $0{\cdot}0833\,a$ down from the top edge
that is $0{\cdot}4167\,a$ above the centre line of the section.

Total compression on concrete $= \frac{1}{2}\,b\,na\,f_c$

$$= \tfrac{1}{2}\,b\,.\,(0{\cdot}25a)\,(0{\cdot}0256 f_s) = 0{\cdot}0032\,abf_s$$

Compression on $A_s' = A_s' f_s' = \frac{2\,ab}{100} \times 0{\cdot}215\,f_s = 0{\cdot}0043\,abf_s$

$$\text{Tension on } A_s = A_s f_s = \frac{2\,ab}{100} f_s = 0{\cdot}020\, ab f_s$$

$$\begin{aligned}\text{Unbalanced tension} &= a b f_s (0{\cdot}020 - (0{\cdot}0032 + 0{\cdot}0043))\\ &= 0{\cdot}0125\, ab f_s\\ &= T.\end{aligned}$$

Taking moments about an axis through the centre of the section

$$\text{Compression on concrete} \times 0{\cdot}4167\, a + A_s' f_s' (0{\cdot}4\, a) + A_s f_s (0{\cdot}4\, a) = T \times e$$

$$\begin{aligned}\text{or } T\,.\,e &= 0{\cdot}0032\, ab f_s \times 0{\cdot}4167\, a + 0{\cdot}0043\, ab f_s \times 0{\cdot}4\, a\\ &\quad + 0{\cdot}020\, ab\, f_s \times 0{\cdot}4\, a\\ &= 0{\cdot}01105\, a^2 b f_s\end{aligned}$$

$$\text{and } e = \frac{0{\cdot}01105\, a^2 b\, f_s}{0{\cdot}0125 ab\, f_s} = 0{\cdot}885\, a$$

$$\frac{e}{a} = 0{\cdot}885$$

We thus have the corresponding values

$$n = 0{\cdot}25$$

$$p = 4\%$$

$$\frac{T}{abf_s} = 0{\cdot}0125$$

$$\frac{e}{a} = 0{\cdot}885$$

This gives one point on Table T I. Other values of *n* and *p* give the whole table.

Example. Case I.—(Note. Cases of combined tension and bending do not often occur except in tanks where the direct tension is small as a rule, and in hopper bottoms which are discussed specially.)

A tension member carries a central pull of 60,000 lbs. and a bending moment of + 100,000 lbs.-ins. at its critical section. The member is 7 ins. wide and 18 ins. deep overall, reinforced with two bars at the top and 2 bars at the bottom, the bars being 14 ins. apart. If f_s is not to exceed 16,000 what is the minimum size of bar?

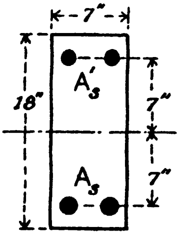


Fig. 44.

This is obviously Case I as $e = \frac{10''}{6}$. For bottom bars take moments about line of top bars

$$A_s \times 16{,}000 \times 14'' = 60{,}000 \times 7'' + 100{,}000$$

$$A_s = 2{\cdot}32 \text{ sq. ins.} = 2 \text{ bars } 1\tfrac{1}{4}'' \text{ diam.}$$

Similarly for A_s'

$$A_s' \times 16{,}000 \times 14'' = 60{,}000 \times 7'' - 100{,}000$$

$$A_s' = 1{\cdot}43 \text{ sq. ins.} = 2 \text{ bars } 1'' \text{ diam.}$$

Example. Case II (a).—A tie carries a tension of 50,000 lbs. and a moment of ± 500,000 lbs.-ins. It is 10 ins. × 18 ins. in section. What steel is required if f_s is not to exceed 18,000 lbs. per sq. in., f_c is not to exceed 750 and m is to be taken as 18?

A symmetrical section is clearly required.

Although Table T I is calibrated for $m = 15$, only a small percentage error is incurred by using it for $m = 18$.

$$T = 50{,}000 \text{ lbs.}$$

$$\frac{T}{abf_s} = \frac{50{,}000}{10 \times 18 \times 18{,}000} = 0{\cdot}0154$$

$$e = \frac{500{,}000}{50{,}000} = 10''$$

$$\frac{e}{a} = \frac{10}{18} = 0{\cdot}555$$

From Table T I $p = 3{\cdot}65\% = \frac{3{\cdot}65 \times 10 \times 18}{100} = 6{\cdot}6$ sq. ins.

That is 3·3 sq. ins. top and bottom.

Say 2 bars 1⅛" plus 2 bars 1" as in Fig. 45.

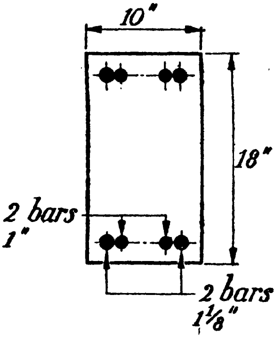


Fig. 45

$$\text{Also } n = 0{\cdot}165$$

$$\therefore f_c = \frac{1}{18} \times 18{,}000 \times \frac{0{\cdot}165}{0{\cdot}735} = 225 \text{ lbs. per sq. in.}$$

Examples. Case II (b).—A rectangular section has to resist a central tension of 50,000 lbs. and a moment of 1,500,000 lbs.-ins. If f_c and f_s are not to exceed 750 and 18,000 lbs. per sq. in and $m = 18$, find a section.

Remembering that when we transfer T to the line of A_s we reduce the effective moment we may start with a section 20 ins. deep overall. If we use four bars in a row it should be about 13 ins. wide.

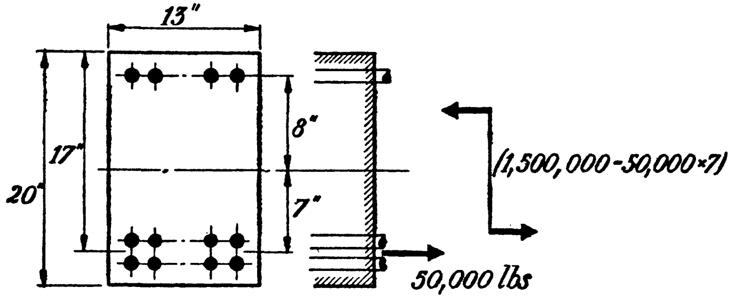


Fig. 46.

Transferring T to the line of A_s the moment becomes

$$(1{,}500{,}000 - 350{,}000) \text{ lbs.-ins. as in Fig. 46.}$$
$$= 1{,}150{,}000$$
$$bd = 13 \times 17 = 221 \text{ sq. ins.}$$
$$bd^2 = 13 \times 17^2 = 3760$$
$$R = \frac{1{,}150{,}000}{3760} = 305$$
$$\frac{R}{f_c} = \frac{305}{750} = 0{\cdot}407$$

As we have to add steel for A_s it is best to aim at making f_s equal to 18,000 which will give $n = 0{\cdot}424$. From Table RB VI, if we follow the line for $n = 0{\cdot}424$ until it crosses the line $\frac{R}{f_c} = 0{\cdot}407$, we find

$$A_s = 1{\cdot}95\%$$
$$A_s' = \text{about } 0{\cdot}95\, A_s$$

Giving $A_s = 1{\cdot}95\%$ of 221 or 4·3 sq. ins.

$A_s' = 0{\cdot}95 \times 4{\cdot}3 = 4{\cdot}08$ sq. ins.

Since $f_s = 18{,}000$

$$\text{Additional } A_s \text{ for } T = \frac{50{,}000}{18{,}000} = 2{\cdot}78 \text{ sq. ins.}$$

Total $A_s = 4{\cdot}3 + 2{\cdot}78 = 7{\cdot}08$ sq. ins.

We may use 4 bars $1\frac{1}{8}$ ins. for A_s'
and 4 bars $1\frac{1}{8}$ ins. plus 4 bars 1 in. for A_s

A wall slab carries a moment of 55,000 lbs.-ins. and a tension of 3500 lbs. per 12 in. width. If we want to avoid compression steel and f_c and f_s must not exceed 900 and 16,000 lbs. per sq. in. respectively with $m = 15$, find a section allowing ¾-in. cover.

With these stresses $n = 0{\cdot}458$

$jd = 0{\cdot}847\, d$

$R = 175\, bd^2$ (see Chapter II).

Neglecting the tension and taking the moment alone, we should require a minimum d of 5·11 ins.

Transferring T to A_s will reduce M so we may try a slab 6 ins. thick overall with $d = 5{\cdot}0$ ins. as Fig. 47.

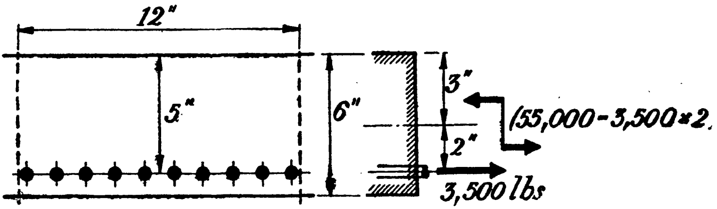


Fig. 47.

$$\text{Modified } M = 55{,}000 - 3500 \times 2 = 48{,}000 \text{ lbs.-ins.}$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{48{,}000}{175 \times 12}} = 4{\cdot}78 \text{ ins.}$$

$$A_s \text{ for modified } M \text{ alone} = \frac{48{,}000}{16{,}000 \times 0{\cdot}847 \times 5''} = 0{\cdot}71 \text{ sq. ins.}$$

$$A_s \text{ for } T \text{ alone} = \frac{3500}{16{,}000} = 0{\cdot}218 \text{ sq. ins.} \quad \text{Total} = 0{\cdot}928 \text{ sq. ins.}$$

We may use ½-in. bars at 2½ ins. centres or use ⅝-in. bars at 4 ins. centres and reduce d by a small fraction. It might, of course, be cheaper and more convenient to use a thicker slab and less steel.

Checking Existing Sections.—If called upon to check the strength of an existing section which is neither Case I nor covered by Table T I, we may commence by transferring the tension T to the line of the tensile steel. The neutral axis in most cases is between 0·2 a and 0·45 a down and we may assume that the centre of pressure is 0·1 a to 0·15 a down from the top edge. This means that the lever arm of the section is 0·75 a to 0·80 a. Taking 0·78 a as a first approximation we can find f_s very closely by taking moments. We may then reduce the actual area of A_s by an amount $\frac{T}{f_s}$ and check the section with the reduced amount of A_s to take the moment alone.

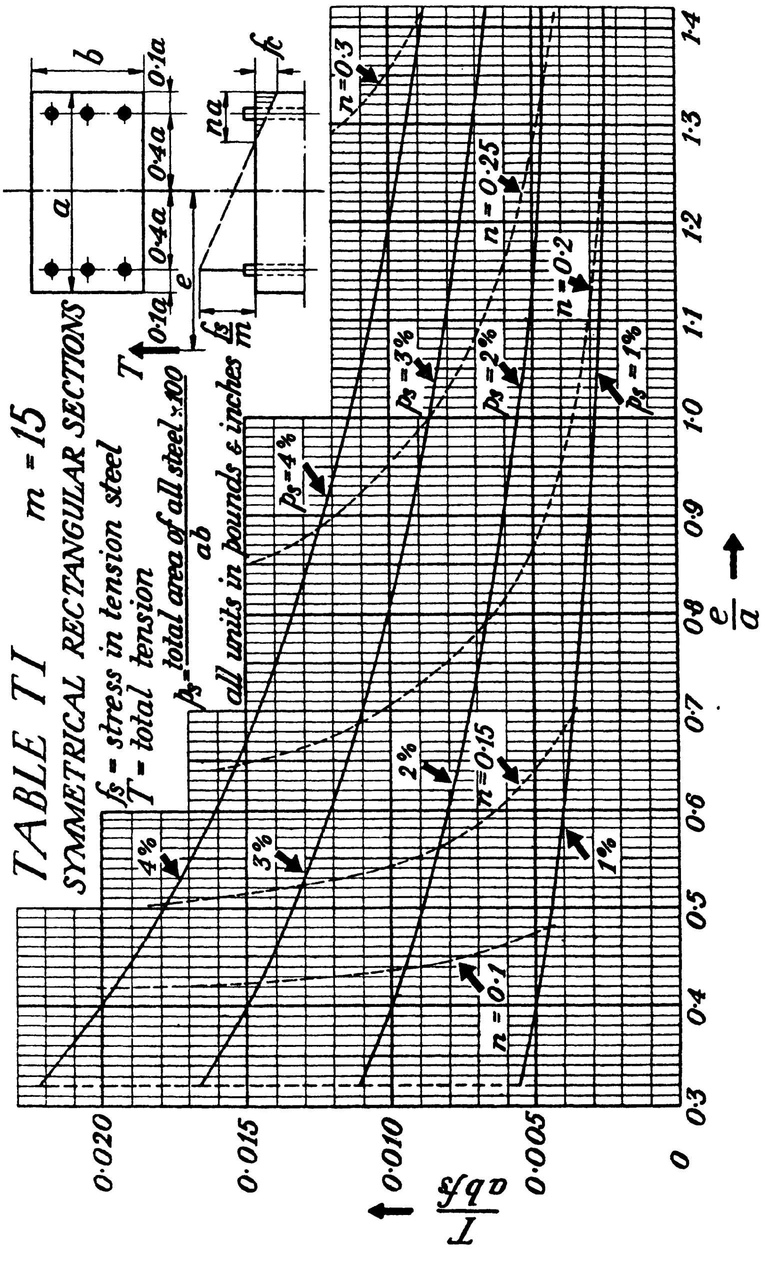


TABLE T1 m = 15

SYMMETRICAL RECTANGULAR SECTIONS

CHAPTER VIII

SECTIONS UNDER BENDING AND COMPRESSION COMBINED

A bending moment alone produces compression over part of the section and tension over the rest. A compressive thrust applied at the centroid produces uniform compressive stress. Sections under combined direct and bending loads show varying stresses, the amount of variation depending on the proportion of direct load to bending moment. As mentioned in the last chapter a central direct load L and a bending moment M can be transformed at once into a direct thrust L acting at a distance e from the centre where $e = \frac{M}{L}$.

In all the analysis that follows e has been taken as the distance from the centre of the *concrete* section. In sections unsymmetrically reinforced this is not the centroid of the whole section when the steel is taken into account.

In practice the great majority of cases of compression and bending are such that the member may be called upon to carry either

(a) direct load alone
(b) bending alone
(c) both.

For example the columns supporting a bunker (or water tower) may carry

(a) direct load alone when the bunker is full and no wind is blowing.

(b) bending moment alone (except for the dead weight of the structure itself which is sometimes negatived by uplift due to wind) when the bunker is empty and a strong wind is blowing.

(c) Any possible combination of the above two cases.

It follows therefore that most sections are symmetrical sections having the same amount of reinforcement top and bottom.

It will be found convenient to divide the analysis into two main parts depending on the ratio of e the eccentricity to a the depth of the section thus :—

First Case. For small values of $\frac{e}{a}$ where no tension whatever is produced.

Second Case (a). For moderate values of $\frac{e}{a}$.

Second Case (b). For large values of $\frac{e}{a}$, i.e., when the section carries a large moment and a relatively small thrust or when a symmetrical solution is not required.

First Case. No Tension Produced.—Let Fig. 48 represent a section carrying a thrust L which acts at a distance e from the centre line of the concrete section.

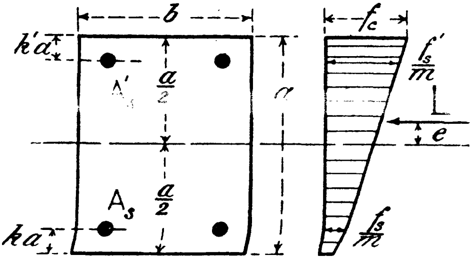


Fig. 48.

The section is reinforced as shown with :—

A_s' near the more heavily compressed edge
A_s near the less heavily compressed edge.

The maximum stress on the concrete is f_c.

Since no tension results, the variation of stress is uniform over the whole concrete section and we can follow the line of reasoning which applies to a steel section under similar loading namely :—

$$\text{maximum stress } f_c = \frac{L}{(\text{area})} + \frac{L\,(e \pm \triangle)}{(\text{modulus of section})}$$

where $\triangle$ is the distance of the centroid of the section from the centre line of the concrete.

Since the addition of steel in compression increases the effective area by $(m - 1)$ times the area of steel introduced

$$\text{we have ; effective area} = ab + (m - 1)\,(A_s + A_s').$$

In order to find the centroid of the section and thence calculate the modulus, the values of k, k', A_s, A_s' and m must be known.

For an unsymmetrical case this method necessitates assuming a section and then checking the value of f_c by a somewhat laborious arithmetical process. This may be necessary on occasions but the author has never met with a case in practice.

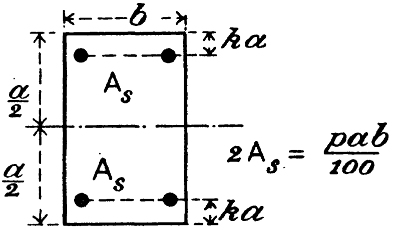


Fig. 49.

The reader may assume that all such cases in practice have :

$$ka = k'a$$
$$A_s = A_s'$$

thus the centroid is at the centre line as shown in Fig. 49 giving

$$\text{Area} = ab + (m - 1)\,(A_s + A_s)$$

Moment of Inertia of Concrete

$$= \frac{1}{12} \cdot ba^3.$$

Moment of Inertia of Steel

$$= A_s \times \left(\frac{a}{2} - ka\right)^2 \times 2\,(m - 1).$$

Total moment of Inertia

$$= \frac{1}{12}\,ba^3 + A_s\left(\frac{a}{2} - ka\right)^2 \times 2\,(m - 1).$$

If Total Steel in section

$$= \frac{p\,.\,ab}{100} = 2\,A_s$$

we have

$$I = \frac{1}{12}ba^3 + \frac{pab}{100} \times a^2\left(\frac{1}{2} - k\right)^2 \times (m - 1)$$

$$= a^3b\left[\frac{1}{12} + \frac{p}{100}\left(\frac{1}{2} - k\right)^2 \times (m - 1)\right].$$

The value of k should in theory vary with the amount of cover and the size, number and arrangement of bars.

In Tables C II (*a*), C II (*b*), C II (*c*), C III (*a*), C III (*b*), C III (*c*) k has been taken as 0·1. In the remaining tables k has been taken as 0·125. Although a value of 0·1 may seem small, the strengthening action of the added steel is probably more than indicated by theory.

Tables C II and C III (with $m = 15$, $k = 0{\cdot}125$) may be compared directly with Tables C II (*a*) and C III (*a*) (with $m = 15$, $k = 0{\cdot}10$) to show the effect of varying k.

Taking a case from Table C II (*a*) putting $m = 15$ and $k = 0{\cdot}1$

$$\text{Moment of inertia} = a^3b\left[\frac{1}{12} + \frac{p}{100}\left(\frac{1}{2} - 0{\cdot}1\right)^2 \times 14\right]$$

$$= a^3b\,[0{\cdot}083 + 0{\cdot}0224\,p]$$

$$\text{Modulus} = a^2b\,[0{\cdot}167 + 0{\cdot}0448\,p]$$

$$\text{Equivalent Concrete Area} = ab + (m - 1)\,(A_s + A_s)$$

$$= ab\,(1 + 0{\cdot}14\,p).$$

$$\text{Therefore } f_c = \frac{L}{ab\,(1 + 0{\cdot}14\,p)} + \frac{Le}{a^2b\,(0{\cdot}167 + 0{\cdot}0448\,p)}$$

$$\frac{L}{abf_c}\left[\frac{1}{(1 + 0{\cdot}14\,p)} + \frac{e}{a\,(0{\cdot}167 + 0{\cdot}0448\,p)}\right] = 1$$

we assume values for $\frac{e}{a}$ and p we can arrive at corresponding values of $\frac{L}{abf_c}$

Thus if $p = 1\%$ and $\frac{e}{a} = 0{\cdot}1$

$$\frac{L}{abf_c}\left[\frac{1}{1{\cdot}14} + \frac{1}{2{\cdot}118}\right] = 1$$

$$\text{giving } \frac{L}{abf_c} = 0{\cdot}74$$

or
$$\frac{L}{abf_c} \times 600 = 445$$

giving a point on Table C II (*a*).

Construction of Tables C II, C II (a), C II (b), C II (c).—Taking the case of Table C II (*a*) where $m = 15$ we may use the analysis given above.

$$\begin{array}{lcllll}
\text{If } p & = & 1\% & 2\% & 3\% & 4\% \\
\text{“ area ”} & = & 1{\cdot}14\,ab & 1{\cdot}28\,ab & 1{\cdot}42\,ab & 1{\cdot}56\,ab \\
\text{Modulus} & = & 0{\cdot}2118\,a^2b & 0{\cdot}2568\,a^2b & 0{\cdot}3015\,a^2b & 0{\cdot}3465\,a^2b
\end{array}$$

$$\begin{array}{lcl}
\text{Suppose max}^{\text{m}}\text{. stress} & = & f_c \\
\text{min}^{\text{m}}\text{. stress} & = & 0{\cdot}5\,f_c
\end{array}$$

then the average stress $= 0{\cdot}75\,f_c$

$$\begin{array}{lcllll}
\text{Total load} & = & 0{\cdot}855\,abf_c & 0{\cdot}96\,abf_c & 1{\cdot}07\,abf_c & 1{\cdot}17\,abf_c \\
\frac{L}{ab} \times \frac{600}{f_c} & = & 512 & 576 & 640 & 700
\end{array}$$

Stress due to bending moment alone $= \frac{1}{2}(f_c - 0{\cdot}5\,f_c) = 0{\cdot}25\,f_c$

Bending moment = stress × modulus

$$= 0{\cdot}0527\,a^2bf_c \quad 0{\cdot}064\,a^2bf_c \quad 0{\cdot}0752\,a^2bf_c \quad 0{\cdot}0865\,a^2bf_c$$

$$e = \frac{\text{Bending Moment}}{\text{Total Load}}$$

$$= 0{\cdot}0618\,a \quad 0{\cdot}0668\,a \quad 0{\cdot}072\,a \quad 0{\cdot}0739\,a$$

Thus giving four separate points on Table C II (*a*).

If we write min^{m}. stress $= f_c$ (central load)
$= 0{\cdot}8\,f_c$
$= 0{\cdot}6\,f_c$ etc.
$= 0$ (tension commences)

we soon have all the values necessary to complete Table C II (*a*).

Tables C II (*b*) and C II (*c*) are calibrated in terms of $\frac{L}{abf_c}$ but Table C II (*a*) is set out in terms of $\frac{L}{abf_c} \times 600$.

This is an example of a table which will give readings for all values of f_c but which is specially arranged to read directly for $f_c = 600$ lbs. per sq. in.

Second Case (a). Tension Occurs.—Let Fig. 50 represent a section under an eccentric load L such that tension is produced. Let the neutral axis occur at a depth na. Let f_c represent the maximum concrete stress.

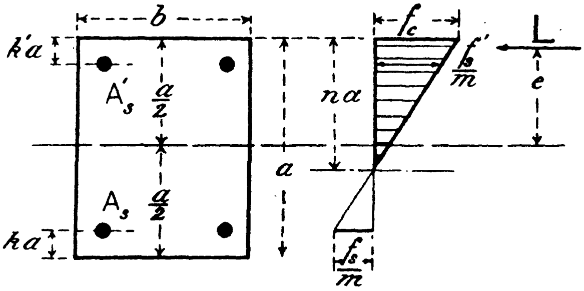


Fig. 50.

We have

$$f_s = \frac{(a - ka - na)}{na} \times f_c \times m$$

$$f_s' = \frac{(na - k'a)}{na} \times f_c \times m$$

$$\text{effective } f_s' = \frac{(na - k'a)}{na} \times f_c \times (m - 1).$$

Total compression on concrete

$$= bna \times \frac{f_c}{2}.$$

Total compression on A_s'

$$= \frac{A_s' \times (na - k'a) \times f_c \times (m - 1)}{na}.$$

Total tension on A_s

$$= A_s \times \frac{(a - ka - na)}{na} \times mf_c.$$

Distance of centroid of concrete stressed area from centre line of section

$$= \left(\frac{a}{2} - \frac{na}{3}\right)$$

$$\text{distance of } A_s = \left(\frac{a}{2} - ka\right)$$

$$\text{,, ,, } A'_s = \left(\frac{a}{2} - k'a\right).$$

Total Load $= L$

$$= \frac{bnaf_c}{2} + \frac{A_s'(na - k'a)f_c(m - 1)}{na} - \frac{A_s(a - ka - na)mf_c}{na}.$$

Taking moments about the centre line

$$Le = \frac{bnaf_c}{2} \times \left(\frac{a}{2} - \frac{na}{3}\right)$$

$$+ \frac{A_s'(na - k'a) f_c (m-1)}{na} \times \left(\frac{a}{2} - k'a\right)$$

$$+ \frac{A_s (a - ka - na)\, m f_c}{na} \times \left(\frac{a}{2} - ka\right).$$

By dividing one expression by the other we obtain a value of $\frac{e}{a}$.

Since it is not possible to apply these expressions in practice we must again have recourse to some form of tabulation.

It is not practicable to tabulate all possible values even if such a course were desirable.

We shall again assume

$$A_s = A_s'$$
$$ka = k'a = 0{\cdot}10\, a.$$

Construction of Table C III (a).—Let us take a section as in Fig. 50 and assume $n = 0{\cdot}6$ with $p = 2\%$ and $m = 15$.

That is $A_s' = A_s = 1\% = \frac{ab}{100}$

Following the analysis above

$$f_s = 7{\cdot}5 f_c$$
$$\text{effective } f_s' = 11{\cdot}7 f_c$$

Total compression on concrete $= 0{\cdot}3\, abf_c$

Total compression on $A_s' = A_s'f_s' = 0{\cdot}117\, abf_c$

Total tension on $A_s = A_s f_s = 0{\cdot}075\, abf_c$

Distance of centroid of concrete stressed area from centre line of section $= 0{\cdot}3\, a$

distance of $A_s = 0{\cdot}4\, a$

$A_s' = 0{\cdot}4\, a$

Total load $=$ nett compression

$= 0{\cdot}3\, abf_c + 0{\cdot}117\, abf_c - 0{\cdot}075\, abf_c = 0{\cdot}342\, abf_c$

Taking moments about the centre line

$$(\text{Total Load}) \times e = 0{\cdot}3\, abf_c \times 0{\cdot}3\, a + 0{\cdot}117\, abf_c \times 0{\cdot}4a + 0{\cdot}075\, abf_c \times 0{\cdot}4\, a$$

$$= 0{\cdot}1668\, a^2bf_c$$

Thus $\frac{L}{abf_c} = 0{\cdot}342$ or $\frac{L}{ab} \times \frac{600}{f_c} = 205$

$$e = \frac{0{\cdot}1668\, a^2bf_c}{0{\cdot}342\, abf_c} \text{ or } \frac{e}{a} = 0{\cdot}487$$

These two values give one point on Table C III (*a*). By writing down the values in columns of figures we may readily calculate sufficient other points to complete the table. Table C III (*b*) for $m = 12$ and Table C III (*c*) for $m = 18$ are similar in construction.

Application of Tables C II, C II (a), C II (b), C II (c), C III, C III (a), C III (b) and C III (c).—To get some idea of the variation involved suppose we take

$$p = 1\%$$
$$f_c = 900$$
$$m = 15$$

Writing down the values from Tables C II (*a*) and C III (*a*)

If $e = 0$ (central load) $L = 1025\,ab$
$e = 0{\cdot}1\,a$ $L = 670\,ab$
$e = 0{\cdot}2\,a$ $L = 495\,ab$
$e = 0{\cdot}3\,a$ $L = 367\,ab$
$e = 0{\cdot}5\,a$ (on edge) $L = 234\,ab$ with $n = 0{\cdot}51$
$e = 1{\cdot}0\,a$ $L = 123\,ab$ with $n = 0{\cdot}37$
$e = 2{\cdot}0\,a$ $L = 60\,ab$ with $n = 0{\cdot}32$

For the last three values :—

$$e = 0{\cdot}5\,a,\ f_s = \frac{15\ (0{\cdot}9\,a - 0{\cdot}51\,a)}{0{\cdot}51\,a} \times f_c = \frac{15 \times 0{\cdot}39}{0{\cdot}51} \times 900$$
$$= 10{,}300 \text{ lbs. per sq. in.}$$

$$e = 1{\cdot}0\,a,\ f_s = \frac{15\ (0{\cdot}9\,a - 0{\cdot}37\,a)}{0{\cdot}37\,a} \times f_c = \frac{15 \times 0{\cdot}53}{0{\cdot}37} \times 900$$
$$= 19{,}300 \text{ lbs. per sq. in.}$$

$$e = 2{\cdot}0\,a,\ f_s = \frac{15\ (0{\cdot}9\,a - 0{\cdot}32\,a)}{0{\cdot}32\,a} \times f_c = \frac{15 \times 0{\cdot}58}{0{\cdot}32} \times 900$$
$$= 24{,}500 \text{ lbs. per sq. in.}$$

In actual design we are given the load *L* and the bending moment. We can at once calculate *e* and have then to design a section by trial and error.

For equal areas, the section having the highest ratio $\frac{a}{b}$ is most effective, for $\frac{L}{ab}$ remains constant while $\frac{e}{a}$ decreases.

The value of *b* may be limited by the height of the column whose section we are considering or, again, the section may have to resist at other times a bending moment about its other principal axis. For example, the columns supporting a long narrow building may have bending moment due to wind blowing on the large or on the small elevation. We cannot therefore at this stage fix any definite ration for $\frac{a}{b}$ which must be decided in each case according to the exact circumstances.

The process of checking sections presents no difficulties thus :—

A 12 in. × 12 in. column with four 1 in. bars carries a central load of 30,000 lbs. and a moment of 200,000 lbs.-ins. What is the maximum value of f_c if $m = 12$?

$$p = \frac{3{\cdot}14}{12 \times 12} \times 100 = 2{\cdot}17\%$$

$$e = \frac{200{,}000}{30{,}000} = 6{\cdot}67'', \quad \frac{e}{a} = \frac{6{\cdot}67}{12} = 0{\cdot}555$$

From Table C III (*b*) we have $\frac{L}{abf_c} = 0{\cdot}287$

$$\therefore \quad f_c = \frac{30{,}000}{12 \times 12} \times \frac{1}{0{\cdot}287} \quad = \quad \textit{730 lbs. per sq. in.}$$

A column is 10 ins. × 18 ins. overall and has four $1\frac{1}{8}$ in. bars. It carries a load 6 ins. away from the short centre-line and on the long centre-line. If f_c is not to exceed 900 lbs. per sq. in., what is the maximum value of this load if $m = 18$?

$$e = 6'', \frac{e}{a} = \frac{6}{18} = 0{\cdot}333.$$

$$p = \frac{4}{10 \times 18} \times 100 = 2{\cdot}22\%.$$

From Table C III (*c*) $\quad \frac{L}{ab\,f_c} = 0{\cdot}525$

that is $\quad \frac{L}{10 \times 18 \times 900} = 0{\cdot}525 \quad$ or $\quad L = $ *85,000 lbs.*

If the load is on the short axis and 6 ins. away from the long axis what is L in the above ?

$$(b = 18'', a = 10'') \quad \frac{e}{a} = \frac{6}{10} = 0{\cdot}6$$

From Table C III (*c*) $\quad \frac{L}{ab\,f_c} = 0{\cdot}328$ or $L = $ *53,000 lbs. only.*

Failure by Tensile Stress.—As the value $\frac{e}{a}$ increases, the total loading approximates more and more to a bending moment pure and simple. The position of the neutral axis is shown on Tables C III (*a*), C III (*b*), and C III (*c*), and we can therefore calculate the value of f_s from the value of f_c thus.

A column is 18 ins. × 18 ins. with four $1\frac{1}{8}$ in. bars. It carries a load of 70,000 lbs. and a moment of 700,000 lbs.-ins. If $m = 15$ what are the values of f_c and f_s ?

$$p = \frac{4}{18 \times 18} \times 100 = 1{\cdot}23\%$$

$$e = \frac{700{,}000}{70{,}000} = 10'', \quad \frac{e}{a} = \frac{10}{18} = 0{\cdot}555.$$

From Table C III (*a*) $\frac{L}{ab} \times \frac{600}{f_c} = 150$ and $n = 0{\cdot}5$

$$f_c = \frac{L}{ab} \times \frac{600}{150} = \frac{70{,}000}{18 \times 18} \times \frac{600}{150} = 862 \textit{ lbs. per sq. in.}$$

$$f_s = f_c \times 15 \times \frac{0{\cdot}4\,a}{0{\cdot}5\,a} = 862 \times 15 \times \frac{4}{5} = 10{,}400 \textit{ lbs. per sq. in.}$$

Second Case (b) for higher values of $\frac{e}{a}$ or for cases where equal steel top and bottom is not required.

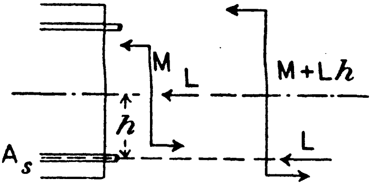


Fig. 51.

Let Fig. 51 represent a section under a moment *M* and a load *L* (central).

Replace *L* (central) by a load *L* acting along the centre-line of the tensile steel and a moment *Lh*. The total moment is now $(M + Lh)$.

Design the section to resist this moment alone disregarding the load *L*. Work out the values of n, A_s and f_s under the bending moment only (see Chapter III). The total compression on the concrete and on A_s' (if any top steel is used) = total tension = $A_s f_s$. Also the moment of the stresses in the concrete and compression steel about the centre-line of the tensile steel = $M + Lh$.

Now decrease A_s by an amount $\frac{L}{f_s}$ so that finally :—

Area of tensile steel $= A_s - \frac{L}{f_s}$.

This amended section will resist the combined stresses.

The compression stresses all remain as before and their moment about the centre-line of $A_s = M + L\,h$. Also the sum of these compressions $= A_s f_s$

$$\text{Total tension} \quad = f_s \left[A_s - \frac{L}{f_s}\right] = A_s f_s - L$$

$\therefore$ Nett thrust on section = *L* (as required).

In a similar way we could transfer the thrust to the centre-line of A_s', design the section for the decreased moment and then increase A_s' by an amount $\frac{L}{f_s'}$.

These methods (or even a combination of them) can be applied in any case that arises but are somewhat indirect and uncertain for large thrusts. In addition to this the final ratio between A_s and A_s' is unknown until the example is fully worked out. If we wished to make $A_s = A_s'$ as a final result, much trial and error would be needed by this method.

An example will illustrate the method. A section is 12 ins. × 7 ins. effective depth. It carries a bending moment of 80,000 lbs.-ins. and a direct load of 4000 lbs. applied at a distance of 4 ins. from the top (compression) edge, stresses not to exceed 750 and 18,000 lbs. per sq. in. with $m = 18$. No compression steel is to be used. (See Fig. 52.)

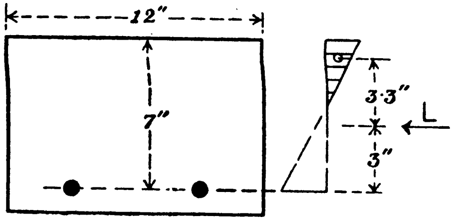


Fig. 52.

Transfer the load to the centre of the tensile steel. The increased moment becomes

$$80{,}000 + 4000 \times 3'' = 92{,}000 \text{ lbs.-ins.}$$

$$R = \frac{92{,}000}{12 \times 7^2} = 157$$

$$\frac{R}{f_c} = \frac{157}{750} = 0{\cdot}208$$

From Table RB VI. $A_s = 1{\cdot}35\%$ and $n = 0{\cdot}49$.

$$A_s = \frac{1{\cdot}35 \times 7 \times 12}{100} = 1{\cdot}14 \text{ sq. ins.}$$

$$f_s = 18 \times 750 \times \frac{0{\cdot}51}{0{\cdot}49} = 14{,}000 \text{ lbs. per sq. in.}$$

$$\therefore \text{ Final area of tensile steel} = 1{\cdot}14 - \frac{4000}{14{,}000} = \mathit{0{\cdot}855\ sq.\ ins.}$$

In the above example transfer the load to compression steel which is placed 0·7 in. down from the top (starting with a section where $A_s' = 0$).

$$\text{New } M = 80{,}000 - 4000 \times 3{\cdot}3'' = 66{,}800 \text{ lbs.-ins.}$$

$$R = \frac{66{,}800}{12 \times 7^2} = 114$$

This is a small value of R for stresses of 750 and 18,000 and we shall have failure by steel stress.

From Fig. 5, A_s will be about 0·74% and from Fig. 2 n will be very nearly 0·4 with $f_s = 18{,}000$

$$A_s = \frac{0 \cdot 74}{100} \times 84 = \mathit{0 \cdot 62\ sq.\ ins.}$$

$$nd = 0 \cdot 4 \times 7 = 2 \cdot 8''$$

$$\text{Effective } f_s' = \frac{2 \cdot 1}{2 \cdot 8} \times 17 f_c = \frac{2 \cdot 1}{2 \cdot 8} \times 17 \times \frac{0 \cdot 4}{0 \cdot 6} \times \frac{18{,}000}{18}$$

$$= 8500 \text{ lbs. per sq. in.}$$

∴ A_s' (which has to resist the direct thrust only)

$$= \frac{4000}{8500} = \mathit{0 \cdot 471\ sq.\ ins.}$$

Comparing these two examples we see that the first has less total steel area than the second. This is due to the small values of R in both cases. Suppose in the above example that the moment is increased to 160,000 lbs.-ins., while the thrust is increased to 8000 lbs.

When using Table RB VI we must remember that stresses of 750 and 18,000 place the value of n at 0·428, and if we fall below this value we shall have f_s more than 18,000 if f_c is 750.

If we transfer the load to the tensile steel Lh = 24,000 lbs.-ins.

$$R = \frac{184{,}000}{12 \times 7^2} = 314 \text{ and } \frac{R}{f_c} = \frac{314}{750} = 0 \cdot 416.$$

From Table RB VI $A_s' = A_s = 2\%$ and $n = 0 \cdot 43$ will meet the case

$$f_s = \frac{0 \cdot 57}{0 \cdot 43} \times 18 \times 750 = 17{,}800 \text{ lbs. per sq. in.}$$

$$\text{Reduction in tensile steel} = \frac{8000}{17{,}800} = 0 \cdot 45 \text{ sq. ins.}$$

Final section $A_s' = 2\% = 1 \cdot 68$ sq. ins.
$A_s = 2\% - 0 \cdot 45 = 1 \cdot 23$ sq. ins.

Total $= 2 \cdot 91$ sq. ins.

Transferring the load to the line of compression steel

$$R = \frac{160{,}000 - 26{,}400}{12 \times 7^2} = 228$$

$$\frac{R}{f_c} = \frac{228}{750} = 0 \cdot 305$$

We may choose $A_s = 1 \cdot 75\%$ and $A_s' = \tfrac{1}{2} A_s = 0 \cdot 875\%$
with $n = 0 \cdot 47$

$$nd = 0 \cdot 47 \times 7 = 3 \cdot 29 \text{ ins.}$$

$$\text{Effective } f_s' = \frac{2{\cdot}59}{3{\cdot}29} \times 17 \times 750 = 10{,}000 \text{ lbs. per sq. in.}$$

$$\text{Additional } A_s' = \frac{8000}{10{,}000} = 0{\cdot}8 \text{ sq. in.}$$

$$\text{Final section } A_s = 1{\cdot}75\% \qquad = 1{\cdot}47 \text{ sq. ins.}$$

$$A_s' = 0{\cdot}875\% + 0{\cdot}8 = 0{\cdot}735 + 0{\cdot}8 = \underline{1{\cdot}535}$$

$$3{\cdot}005 \text{ sq. ins.}$$

It will be seen that this Second Case (*b*) admits of any number of solutions depending on the section chosen to resist the moment. As pointed out previously, we can find an infinite number of sections to resist any external force or combination of forces and the best solution is the one that fits in best with the particular case.

Application of Second Case (b).—This case seldom occurs in practice. Most cases of combined load and moment occur in columns which are symmetrically reinforced and where $\frac{e}{a}$ is somewhere between 0 and 0·5. If it should occur its treatment will depend on the value of R.

If R is small increase the tensile steel.

If R is large put steel top and bottom and increase the compression steel.

It seems peculiar at first sight that a compressive force can be met by increasing the tensile steel. The action of such an increase is to lower the neutral axis and thus bring a larger area of concrete into play.

Square Columns Bending about a Diagonal.—Only one case will be considered, i.e., in Fig. 54 when $k = k' = {\cdot}125$ and $m = 15$.

Before attacking the main problem we must solve the following lemma.

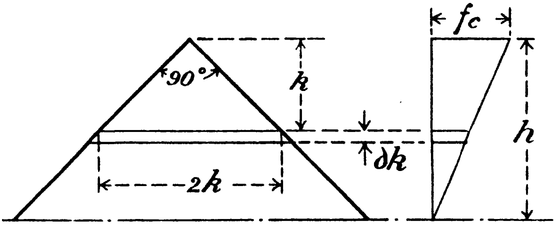


Fig. 53.

Let Fig. 53 represent a triangular area under a stress which varies uniformly from f_c at the point to zero at the base.

We have to find the total load it carries and the line on which this resultant load acts. We will assume that the upper angle is

90° and the total height h. Take a small strip parallel to the base and distant k from the apex. Since the apex angle is 90° the breadth of this strip is $2k$. The height of the strip is δk which is small.

Area of strip $= 2k\,\delta k$

Intensity of stress on strip $= \dfrac{h-k}{h} f_c$

Total stress on strip $= 2k\,\delta k \times \dfrac{h-k}{h} f_c.$

Total stress on whole area

$$= \int_{k=o}^{k=h} 2k\,\delta k \times \frac{h-k}{h} f_c = \frac{h^2}{3} f_c.$$

To find the centre of pressure take moments about a line through the apex parallel to the base.

We have already:

Total pressure on strip $= 2k\,\delta\,k \times \dfrac{h-k}{h} f_c.$

Moment of this pressure about the apex

$$= 2k\,\delta k \times \frac{h-k}{h} f_c \times k.$$

Total moment about apex

$$= \int_{k=o}^{k=h} \frac{2 f_c}{h}(hk^2 - k^3)\,\delta k = \frac{h^3 f_c}{6}$$

$$= (\text{Total pressure}) \times (\text{depth to centre of pressure})$$

$$= \frac{h^2 f_c}{3} \times (\text{depth to centre of pressure})$$

$\therefore$ depth to centre of pressure $= \dfrac{h^3 f_c}{6} \times \dfrac{3}{h^2 f_c} = \dfrac{h}{2}$ i.e., half-way between the apex and base.

Instead of taking a general case we will commence at once on our particular problem.

Assume $p = 1\%$, i.e., area of each bar $= \dfrac{pa^2}{400} = \dfrac{a^2}{400}$. Depth of neutral axis $= \cdot 7\,a$.

Fig. 54 shows the resulting dimensions.

Total pressure on concrete

$$= f_c \times \frac{(\cdot 7\,a)^2}{3} = \cdot 163\,a^2 f_c.$$

Compression on top bar (effective)

$$= \frac{a^2}{400} \times \frac{\cdot 523\,a}{\cdot 7\,a} \times 14 f_c = \cdot 0262\,a^2 f_c.$$

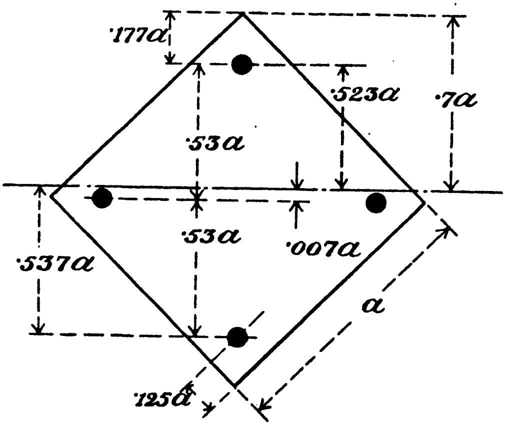


Fig. 54.

Tension on two middle bars

$$= \frac{\cdot 007\,a}{\cdot 7\,a} \times 15 f_c \times \frac{a^2}{200} = \cdot 0007\,a^2 f_c.$$

Tension on bottom bar

$$= \frac{\cdot 537\,a}{\cdot 7a} \times \frac{a^2}{400} \times 15 f_c = \cdot 0288\,a^2 f_c.$$

Moment of total concrete compression about the diagonal

$$= \cdot 163\,a^2 f_c \times (\cdot 007\,a + \cdot 35\,a) = \cdot 0582\,a^3 f_c.$$

Moment of compression on top bar

$$= \cdot 0262\,a^2 f_c \times \cdot 53\,a = \cdot 0139\,a^3 f_c.$$

Moment of two centre bars = 0 (very nearly).

Moment of total tension on bottom bar

$$= \cdot 0288\,a^2 f_c \times \cdot 53\,a = \cdot 0153\,a^3 f_c.$$

Resultant compression on section = L

$$= \cdot 163\,a^2 f_c + \cdot 0262\,a^2 f_c - \cdot 0007\,a^2 f_c - \cdot 0288\,a^2 f_c$$

$$= \cdot 160\,a^2 f_c \quad \text{i.e.} \frac{L}{a^2} \times \frac{600}{f_c} = 96.$$

Resultant moment about diagonal

$$= \cdot 0582\,a^3 f_c + \cdot 0139\,a^3 f_c + 0 + \cdot 0153\,a^3 f_c = \cdot 0874\,a^3 f_c.$$

Eccentricity of resultant pressure

$$= \frac{\cdot 0874\,a^3 f_c}{\cdot 160\,a^2 f_c} = \cdot 546\,a.$$

Thus we have for $p = 1\%$ on Table C V

$$\frac{L}{a^2} \times \frac{600}{f_c} = 96$$

$$\frac{e}{a} = \cdot 546$$

Giving one point on Table C V.

Construction of Tables C IV and C V.—Table C IV which corresponds to Table C II is worked out by writing down the modulus of the section as previously explained in the early part of this chapter for Table C II. Table C V is arrived at by extending the example quoted above.

It will be seen at once that a square column is much weaker when bent about its diagonal.

For example writing $a = b$ and $p = 2\%$.

Table C III if $e = \cdot 5\,a$, $\frac{L}{a^2} \times \frac{600}{f_c} = 188$ lbs. per sq. in.

Table C V if $e = \cdot 5\,a$, $\frac{L}{a^2} \times \frac{600}{f_c} = 138$,, ,, ,, ,,

A reduction in carrying capacity of 26%.

The application of these two tables is sufficiently obvious being exactly similar to that of Tables C II, C II(*a*), C III, and C III (*a*).

Circular Column Sections under Eccentric Load.—In a circular column the number of bars—6, 8, 10, 12, etc.—will vary with the diameter of the column. For purposes of analysis, instead of treating each bar separately, the steel will be assumed as consisting of a thin circular shell. Thus in Fig. 55

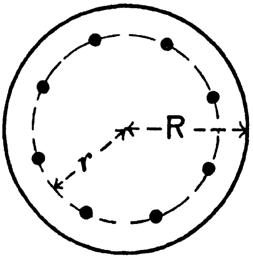


Fig. 55.

The total area of these eight bars will be assumed as spread along the circumference of a circular shell whose radius is equal to *r*.

Fig. 56 shows the resulting arrangement.

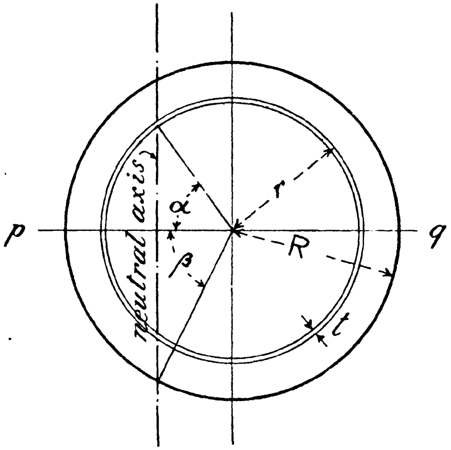


Fig. 56.

Analysis.—Let Fig. 56 represent a circular section whose radius $= R$ reinforced with a thin shell of steel whose radius $= r$ and thickness $= t$.

Area of section of shell $= 2\ \pi r\ t$

percentage area of steel $= \dfrac{2\ \pi r\ t}{\pi\ R^2} \times 100.$

Let the neutral axis occur where shown, the column being assumed to bend about a centre-line $y - y$.

We have as before

Total thrust on section $= L$

= Total compression on concrete area above neutral axis added to the total compression on steel area above neutral axis less the total tension on the steel area below the neutral axis.

Also $L \times$ (eccentricity of application) $= L\ e$

= Moment of all the above forces about the axis of bending.

To simplify the working, the stresses on the concrete and on the steel will be worked out separately.

Let Fig. 57 represent the stressed concrete area.

Take a thin strip parallel to the neutral axis which subtends an angle $2\ \theta$ at the centre

Length of strip $= 2\ R \sin \theta$

distance from centre line $= R \cos \theta$

width of strip $= \delta\ (R \cos \theta) = R \sin \theta\ \delta\theta$ (positive).

Area $= 2\ R^2 \sin^2 \theta\ \delta\theta.$

Intensity of stress $= f_c \times \dfrac{R\ (\cos \theta - \cos \beta)}{R\ (1 - \cos \beta)}.$

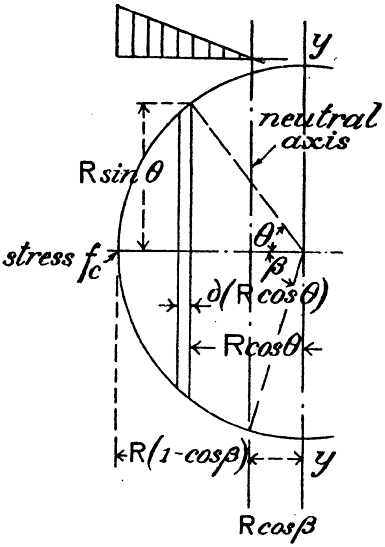


Fig. 57.

Total stress on element

$$= \frac{2 f_c R^3}{R(1-\cos\beta)} (\cos\theta \sin^2\theta - \cos\beta \sin^2\theta)\, d\,\theta.$$

Total compression on concrete

$$= \frac{2 f_c R^2}{(1-\cos\beta)} \int_0^\beta \left(\sin^2\theta \cos\theta - \cos\beta \left(\frac{1-\cos 2\theta}{2}\right)\right) \delta\,\theta$$

$$= \frac{2 f_c R^2}{(1-\cos\beta)} \left[\frac{\sin^3\beta}{3} - \frac{\beta}{2}\cos\beta + \frac{\cos\beta \sin 2\beta}{4}\right].$$

Again we have as above
Total stress on element

$$= \frac{2 f_c R^3}{R(1-\cos\beta)} (\cos\theta \sin^2\theta - \cos\beta \sin^2\theta)\, \delta\theta.$$

Moment of this stress about centre-line = stress × $R \cos\theta$

$$= \frac{2 f_c R^3}{1-\cos\beta} (\cos^2\theta \sin^2\theta - \cos\beta \cos\theta \sin^2\theta)\, \delta\theta$$

$$= \frac{2 f_c R^3}{1-\cos\beta} \left(\frac{(\sin 2\theta)^2}{4} - \cos\beta \cos\theta \sin^2\theta\right) \delta\theta.$$

$$\text{Total moment} = \frac{2 f_c R^3}{1-\cos\beta} \int_0^\beta \left(\frac{1-\cos 4\theta}{8} - \cos\beta \cos\theta \sin^2\theta\right) \delta\theta$$

$$= \frac{2 f_c R^3}{1 - \cos\beta}\left(\frac{\beta}{8} - \frac{\sin 4\beta}{32} - \frac{\cos\beta \sin^3\beta}{3}\right).$$

Coming now to the thin ring of steel we have in Fig. 58 taking the area in compression only, first. Depth of neutral axis as before

$$= R(1 - \cos\beta)$$

$$\text{as } R\cos\beta = r\cos\alpha \text{ we have}$$

$$R(1 - \cos\beta) = R - r\cos\alpha$$

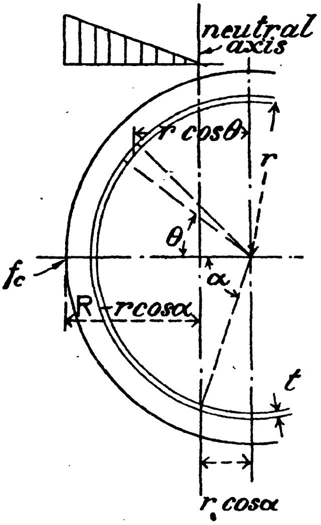


Fig. 58.

Take a small element of steel $r\,\delta\,\theta\,t$.

Intensity of stress on element

$$= f_c \times \frac{(r\cos\theta - r\cos\alpha)}{R - r\cos\alpha} \times (m - 1) \text{ effective.}$$

Total compression on element

$$= f_c \frac{r^2\,\delta\,\theta\,t}{R - r\cos\alpha} (\cos\theta - \cos\alpha)(m - 1).$$

Total compression on compression area of steel

$$= 2\int_0^{\alpha} \frac{f_c r^2 t}{(R - r\cos\alpha)} (\cos\theta\, d\theta - \cos\alpha\, d\theta)(m - 1)$$

$$= \frac{2 f_c (m - 1)}{R - r\cos\alpha} (\sin\alpha - \alpha\cos\alpha)\, r^2 t.$$

Moment of compression on element about centre-line

$$= f_c \times \frac{(r \cos \theta - r \cos \alpha)}{R - r \cos \alpha} \times (m - 1) \times r \delta \theta t \times r \cos \theta.$$

Moment of total compression on whole compression area

$$= 2 \int_{\theta=o}^{\theta=\alpha} \frac{f_c r^3 t}{(R - r \cos \alpha)} [\cos^2 \theta d \theta - \cos \alpha \cos \theta d \theta] (m - 1)$$

$$= \frac{2 f_c (m - 1) r^3 t}{(R - r \cos \alpha)} \left[\frac{\alpha}{2} - \frac{\sin 2 \alpha}{4}\right].$$

Coming to the portion of the steel ring in tension we have by similar processes

$$\text{Total tension} = \frac{2 f_c (m) r^2 t}{(R - r \cos \alpha)} [\sin \alpha + (\pi - \alpha) \cos \alpha].$$

$$\text{Moment of total tension} = \frac{2 f_c m r^3 t}{(R - r \cos \alpha)} \left[\frac{\pi}{2} - \frac{\alpha}{2} + \frac{\sin 2 \alpha}{4}\right].$$

It should be noticed in all the above expressions that $(r \cos \alpha)$ is the distance of the neutral axis from the centre-line and may be greater than r. In such cases the constants and variables can be distinguished by reference to the original expression.

Construction of Tables C VI and C VII.—Taking Table C VI first we shall assume

$$m = 15$$

$$r = \cdot 75 R$$

$$\text{total area of steel} = 2 \pi r t$$

$$\text{area of concrete} = \pi R^2$$

$$\text{percentage area of steel} = \frac{2 \pi r t \times 100}{\pi R^2}$$

$$= \frac{1 \cdot 5 R t \times 100}{R^2}$$

$$\text{or } t = \frac{p R}{150}.$$

By putting in actual values of α, β, and p we can calculate the total load carried and the eccentricity of application.

To take an easy example

$$\text{put } p = 1\% \text{ and } \alpha = 90^\circ = \frac{\pi}{2} \text{ radians.}$$

Then $t = \frac{R}{150}$ and $\beta = 90^\circ$

Total compression on concrete

$$= \frac{2 f_c R^2}{1} \left[\frac{1}{3} - 0 + 0\right] = \cdot 667 f_c R^2$$

compression on compression steel

$$= \frac{2 \times 14}{1}[1-0] \times \frac{9}{16} \times \frac{R^2}{150} \times f_c = \cdot 105 f_c R^2$$

tension on tensile steel

$$= 2 \times 15 \times \frac{9}{16}[1+0] \times \frac{R^2}{150} \times f_c = \cdot 112 f_c R^2.$$

Total compression on section

$$= f_c R^2 [\cdot 667 + \cdot 105 - \cdot 112]$$

$$L = \cdot 660 f_c R^2$$

$$\frac{L}{\pi R^2} \times \frac{600}{f_c} = 127$$

Moment of concrete stress about centre line

$$= \frac{2}{1}\left(\frac{\pi}{16} - 0 - 0\right) R^3 f_c = \cdot 391 R^3 f_c$$

Moment of compression steel

$$= 2 \times 14 \times \frac{27}{64} \times \frac{R^2}{150}\left[\frac{\pi}{4} - 0\right] \times f_c = \cdot 062 f_c R^3$$

Moment of tensile steel

$$= 2 \times 15 \times \frac{27}{64} \times \frac{R^3}{150}\left[\frac{\pi}{2} - \frac{\pi}{4} + 0\right] = \cdot 066 f_c R^3$$

Total moment $= f_c R^3 (\cdot 391 + \cdot 062 + \cdot 066) = \cdot 519 f_c R^3$

$$e = \frac{\cdot 519 f_c R^3}{\cdot 660 f_c R^2} = \cdot 78 R = \cdot 39 D.$$

Thus for $\quad \dfrac{L}{\pi R^2} \times \dfrac{600}{f_c} = \dfrac{4L}{\pi D^2} \times \dfrac{600}{f_c} = 127$

we have $\quad \dfrac{e}{D} = \cdot 39$

giving one point on Table C VII.

By putting in different values of a and p the whole table is worked out.

Coming now to Table C VI where no tension is developed we can proceed as for Table CII, i.e., by writing down the modulus of section we have

Moment of inertia of a circle

$$= \frac{\pi R^4}{4}$$

Moment of inertia of a thin ring

$$= \delta\left(\frac{\pi R^4}{4}\right) = \pi R^3 \delta R.$$

Thus, moment of inertia of concrete

$$= \frac{\pi R^4}{4}$$

of steel ring $= (m - 1)\,\pi\, r^3\, t$ (effective).

Modulus of section $= \frac{1}{R}\left(\frac{\pi R^4}{4} + (m-1)\,\pi\, r^3\, t\right)$

putting $m = 15, \quad r = \frac{3\,R}{4}$ and $t = \frac{p\,R}{150}$

$$\text{Modulus} \quad = \pi\left(\frac{R^3}{4} + \frac{14 \times 27}{64} \times R^3 \times \frac{p}{150}\right)$$

$$= \frac{\pi R^3}{4}\,(1 + \cdot157\,p)$$

" Area " $= \pi R^2\,(1 + \cdot14\,p)$

Let the maximum stress on the section (see Table C VI) be f_c while the minimum is $\cdot5\,f_c$
average stress $= \cdot75\,f_c$, variation $= \cdot5\,f_c$, bending stress $= \cdot25\,f_c$

put $p = 1\%$

$$\therefore \text{ Total load} = L = \cdot75\,f_c \times \pi R^2\,(1\cdot14) = \cdot852\,R^2\,\pi\,f_c$$

$$\text{Bending moment} \quad = \cdot25\,f_c \times \frac{\pi R^3}{4}\,(1\cdot157) = \cdot0721\,R^3\,\pi\,f_c$$

$$\frac{L}{\pi R^2 f_c} \times 600 = \frac{4\,L}{\pi D^2 f_c} \times 600 = 512$$

$$\frac{e}{R} = \frac{\cdot0721}{\cdot852} = \cdot084 \quad \text{i.e.,} \quad \frac{e}{D} = \cdot042$$

giving a point on Table C VI.

By writing $p = 2\%, \ = 3\%$, etc.
and writing minimum stress $= \cdot75\,f_c, \ = \cdot25\,f_c$, etc.
the whole of Table C VI soon results.

Application of Tables C VI and C VII.—These tables are exactly similar to Tables C II to C V in their application. If the section is known then f_c can be found immediately. If the loading is known and the section is required we can by trial and error soon find a suitable section by varying D or p or both. If the size of column is not important p may be taken as from 1% to 2% according to the nearest practical size of bar.

Example.—A section carries a load of 100,000 lbs. (central) and a bending moment of 600,000 lbs.-ins. If f_c is not to exceed 800 find a suitable circular section (assuming $m = 15$). We have

$$e = \frac{600{,}000}{100{,}000} = 6 \text{ ins.}$$

$$\text{Try } D = 18'' \quad \text{then} \frac{e}{D} = \frac{6}{18} = \cdot 333$$

$$\text{Then} \quad \frac{L \times 4}{\pi D^2} = \frac{100{,}000 \times 4}{\pi \times 18 \times 18} = 392 \text{ and since } f_c = 800$$

$$\frac{L \times 4}{\pi D^2} \times \frac{600}{f_c} = 294.$$

Referring to Table C VII the point indicated

for $\frac{4L}{\pi D^2} \times \frac{600}{f_c} = 294$ and $\frac{e}{D} = \cdot 333$

lies above 3% and would be about 6% steel which is excessive.

$$\text{Try } D = 20'' \quad \text{then } \frac{e}{D} = \cdot 3$$

$$\frac{4L}{\pi D^2} \times \frac{600}{f_c} = 239$$

this point lies on the line $p = 3\% = 9{\cdot}5$ sq. ins.

$$\text{Try } D = 22'' \quad \text{then} \frac{e}{D} = \cdot 273$$

$$\frac{4L}{\pi D^2} \times \frac{600}{f_c} = 198 \text{ whence } p = 1{\cdot}3\% = 4{\cdot}95 \text{ sq. ins.}$$

Thus we may use either

(a) a 20″ diam. column with 9·5 sq. ins. of steel
(b) a 22″ ,, ,, ,, 4·95 sq. ins. of steel.

When the reader comes to Chapter XVI he can compare the cost per foot of these two designs.

Hollow circular Column sections. Analysis.—In cases where the thickness of the concrete is about one-tenth or one-twelfth of the diameter as occurs in chimney design we can assume the variation of stress along a radius is small. That is, that the stress on the inside face is very nearly equal to the stress on the outside face at corresponding points and both these stresses may be assumed as equal to the stress at the mean diameter. (This does not refer to temperature stresses.)

Let Fig. 59 represent a thin hollow concrete section reinforced with a ring of bars which can be replaced for analysis by a thin annular ring of steel.

Let t_s = thickness of steel ring
t_c = thickness of concrete
R = mean radius

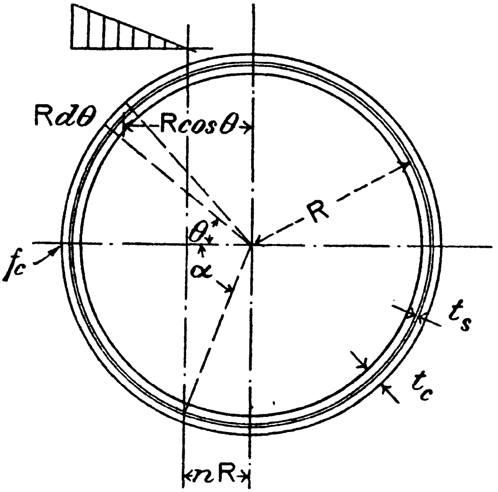


Fig. 59.

The neutral axis occurs at a distance $n\ R$ from the centre line as shown.

Take an element $R\ d\ \theta$ long.

Let f_c = maximum stress on concrete.

Take first of all the concrete area under compression.

Area of concrete element $= R\ d\ \theta\ t_c$.

Intensity of stress on element

$$= f_c \times \frac{(\mathrm{R} \cos \theta - n\ R)}{R - n\ R}.$$

Total compression on element of concrete

$$= R\ d\ \theta\ t_c \times f_c \times \frac{(R \cos \theta - n\ R)}{R - n\ R}$$

Total compression on concrete

$$= 2 \int_0^a t_c f_c R \frac{(\cos \theta - n)}{(1 - n)}\ d\ \theta$$

$$= \frac{2\ t_c f_c\ R}{(1 - n)} (\sin a - n\ a) \text{ where } n = \cos a$$

Effective intensity of stress on steel element

$$= (m - 1) f_c \frac{(\cos \theta - n)}{(1 - n)}$$

and integrating as above.

We have

Total compression on steel ring

$$= \frac{2\,t_s f_c R}{(1-n)}(\sin a - n\,a)\,(m-1).$$

Moment of total compression on concrete element about the centre line

$$= R\,d\,\theta\,t_c \times f_c \times \frac{(\cos\theta - n)}{(1-n)} \times R\cos\theta.$$

Total moment of concrete compression

$$= 2\int_0^a \frac{R^2 t_c f_c}{(1-n)}(\cos^2\theta - n\cos\theta)\,d\,\theta$$

$$= \frac{2\,t_c f_c R^2}{(1-n)}\left(\frac{a}{2} + \frac{\sin 2\,a}{4} - n\sin a\right)$$

similarly :—

Total moment of compression steel

$$= \frac{2\,t_s f_c\,R^2\,(m-1)}{(1-n)}\left(\frac{a}{2} + \frac{\sin 2\,a}{4} - n\sin a\right).$$

If we take the two expressions for the compression steel and write

m for $(m-1)$, $(\pi - \theta)$ for θ, $(\pi - a)$ for a
and $-\,n$ for n (inside bracket).

We have

total tension on tensile steel

$$= \frac{2t_s f_c R}{(1-n)}(\sin a + n\,(\pi - a)) \times m.$$

Total moment of tensile steel

$$= \frac{2\,t_s f_c\,R^2 \,.\, m}{(1-n)}\left(\frac{\pi - a}{2} - \frac{\sin 2a}{4} + n\sin a\right).$$

Construction of Table HC II.—In the above analysis :—

total area of steel $= 2\,\pi\,R\,t_s$

total area of concrete $= 2\,\pi\,R\,t_c$

∴ percentage area of steel

$$= 100 \times \frac{2\,\pi\,R\,t_s}{2\,\pi\,R\,t_c} = \frac{100\,t_s}{t_c} = p.$$

If we write $m = 15$

$p = 1\%, = 2\%, = 3\%$

$n = +\,{\cdot}28, = +\,{\cdot}20, = 0, = -\,{\cdot}20$ etc.

we can construct a table showing the relation between the total load, the eccentricity and the stresses.

Thus if

$$n = +\,{\cdot}20 \text{ and } p = 1\%$$
$$\alpha = 78^\circ\,30' = 1{\cdot}37 \text{ radians}$$
$$\sin\alpha = {\cdot}98, \quad \cos\alpha = {\cdot}20.$$

Total compression on concrete

$$= \frac{2\,t_c f_c\,R}{{\cdot}8}\,({\cdot}98 - {\cdot}2 \times 1{\cdot}37) = 2\,t_c f_c\,R \times {\cdot}88.$$

Total compression on steel

$$= \frac{2\,t_s f_c\,R}{{\cdot}8}\,({\cdot}98 - {\cdot}2 \times 1{\cdot}37) \times 14 = 2\,t_c f_c\,R \times {\cdot}123$$

Total tension on steel

$$= \frac{2\,t_s f_c\,R}{{\cdot}8}\,({\cdot}98 + {\cdot}2(3{\cdot}14 - 1{\cdot}37)) \times 15 = 2\,t_c f_c\,R \times {\cdot}254.$$

Moment of concrete stress

$$= \frac{2t_c f_c\,R^2}{{\cdot}8}\left(\frac{1{\cdot}37}{2} \times \frac{2 \times {\cdot}98 \times {\cdot}20}{4} - {\cdot}20 \times {\cdot}98\right) = 2\,t_c f_c\,R^2 \times {\cdot}732.$$

Moment of compression steel stress

$$= \frac{2\,t_s f_c\,R^2 \times 14}{{\cdot}8}\left(\frac{1{\cdot}37}{2} + \frac{2 \times {\cdot}98 \times {\cdot}20}{4} - {\cdot}20 \times {\cdot}98\right)$$
$$= 2\,t_c f_c\,R^2 \times {\cdot}103.$$

Moment of tensile steel stress

$$= \frac{2\,t_s f_c\,R^2 \times 15}{{\cdot}8}\left(\frac{3{\cdot}14 - 1{\cdot}37}{2} - \frac{2 \times {\cdot}98 \times {\cdot}20}{4} + {\cdot}20 \times {\cdot}98\right)$$
$$= 2\,t_c f_c\,R^2 \times {\cdot}183.$$

Total load on section $= L$

$$= 2\,t_c f_c\,R\,({\cdot}88 + {\cdot}123 - {\cdot}254) = {\cdot}75 \times 2\,t_c f_c\,R$$

Total moment about centre-line $= L \times e$

$$= 2\,t_c f_c\,R^2\,({\cdot}732 + {\cdot}103 + {\cdot}183) = 1{\cdot}018 \times 2\,t_c f_c\,R^2$$

$$\therefore \quad \frac{L}{2\,R\,t_c f_c} = {\cdot}75, \quad \frac{e}{R} = \frac{1{\cdot}018}{{\cdot}75} = 1{\cdot}35 \quad \text{i.e. } \frac{e}{D} = {\cdot}675$$

giving one point on Table HC II (on table neutral axis is $n\,D$—not $n\,R$)

By putting in different values of n and p the whole table may be arranged.

Construction of Table HC I.—If no tension is produced we may write down the modulus of section thus

Moment of Inertia of concrete about a diameter

$$= \pi\,R^3\,t_c.$$

Moment of Inertia of steel

$$= \pi\,R^3\,t_s \times (m - 1)$$
$$\text{if } m = 15$$
$$= \pi\,R^3\,t_c \times {\cdot}14\,p.$$

$$\text{Total modulus} \quad = \frac{\pi R^3 t_c (1 + \cdot 14 p)}{R}$$

$$= \pi R^2 t_c (1 + \cdot 14 p).$$

$$\text{“ Area ”} = 2 \pi R t_c (1 + \cdot 14 p).$$

$$\text{Put } f_c = \text{maximum stress}$$

$$\cdot 5 f_c = \text{minimum stress}$$

$$\text{average stress} = \cdot 75 f_c, \text{ bending stress} = \cdot 25 f_c$$

$$L = 2 \pi R t_c (1 + \cdot 14 p) \times \cdot 75 f_c$$

$$L \times e = \pi R^2 t_c (1 + \cdot 14 p) \times \cdot 25 f_c$$

$$\text{i.e. } \frac{L}{2 R t_c f_c} = \cdot 75 \times \pi \times (1 + \cdot 14 p) = \frac{L}{D t_c f_c}$$

$$e = \frac{\pi R^2 t_c (1 + \cdot 14 p) \times \cdot 25 f_c}{2 \pi R t_c (1 + \cdot 14 p) \times \cdot 75 f_c} = 2R \times \cdot 083$$

$$\text{i.e. } \frac{e}{D} = \cdot 083$$

$$\text{If } p = 0\%, = 1\%, = 2\%, = 3\%$$

$$\text{we have } \frac{L}{D t_c f_c} = 2 \cdot 36, = 2 \cdot 69, = 3 \cdot 03, = 3 \cdot 35$$

giving four points on Table HC I.

Application of Tables HC I and HC II.—The use of these two tables is exactly similar to that of the other tables in this chapter.

They are intended primarily for chimney design in which the tensile stress in the steel may be important. To this end the values of *n* are plotted on Table HC II. There are other stresses on chimneys which will be mentioned later.

Sections not Included in the Tables C II to C VII.—*T-sections* under combined load offer little difficulty. The tensile force is centred near the bottom of the rib while the compressive force acts at or near the centre of the flange. Having thus located the centre of tension and the centre of compression the values of the total tension and total compression are easily arrived at by taking moments. The area of steel in the rib follows immediately while the strength of the flange may be checked from Table TB I.

Polygonal Sections.—Octagonal and hexagonal sections are close enough to circular sections to allow us to use Tables C VI and C VII to solve such cases. The section may be transformed into a circular section of equal area and treated as such.

Rectangular Sections with many Bars.—The standard sections have been worked out for four bars (or four groups of bars) one in each corner.

Fig. 60 shows three symmetrical sections all bending about the axis $x - x$.

Type I offers no difficulty. The bars are all equally effective in resisting bending about this axis and Tables C II and C III or C II*a* and C III*a* or C II*b* and C III*b* or CII*c* and C III*c* will apply.

Types (II) and (III) are usually found only in larger sections. For purely central loads they are just as effective as type (I) or as the standard type. For eccentric loads, however, the bars near the axis of bending $x - x$ lose their efficiency as they carry little stress and resist no moment. These cases may be approximated to by writing :—

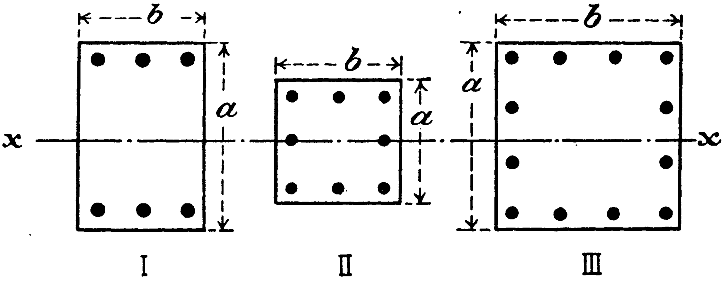


Fig. 60.

$$\text{for type (II)}, \quad p = \frac{\text{area of 7 bars only}}{ab}$$

$$\text{for type (III)}, \quad p = \frac{\text{area of 10 bars only}}{ab}$$

and applying Tables C II and C III or C II*a* and C III*a*, etc.

As an alternate method the bars may be replaced by a thin hollow square as shown in Fig. 61 and values of $\frac{L}{abf_c}$ and $\frac{e}{a}$ may be tabulated.

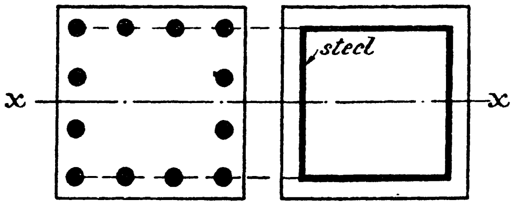


Fig. 61.

Unsymmetrical Sections.—Must be treated under Second Case (*b*) directly. For checking such sections the position of the neutral axis must be found by trial and error. The author has never come across such a case in practice.

Comparison of Shapes to Resist eccentric Load.—If the column is called upon to resist a bending moment about one fixed axis only, a rectangular section will be the obvious choice with the ratio $\frac{a}{b}$ as high as possible.

If however the column may be bent about any axis as for example the columns supporting a water-tower, where the wind may blow from any direction, a symmetrical section will be used.

The choice therefore lies between a square section or a circular section (or hexagon or octagon, etc.).

Let us take as an example,

a 10 ins. × 10 ins. square column, area = 100 sq. ins.
and a circular column 11·25 ins. diam., area = 100 „ „

each having $p = 1\%$ with $m = 15$ in all cases.

These two columns will require the same amount of concrete and steel per foot to construct them.

Let $f_c = 600$

and write $e = 0'', = 1'', = 3'', = 5'', = 10''$

so that $\frac{e}{a} = 0, = \cdot 1, = \cdot 3, = \cdot 5, = 1{\cdot}0$

$\frac{e}{D} = 0, = \cdot 089, = \cdot 267, = \cdot 444, = \cdot 89$

from Tables C II and C III.

The square column on its major axis

for $\frac{e}{a} = 0, = \cdot 1, = \cdot 3, = \cdot 5, = 1{\cdot}0$

has $\frac{L}{100} = 684, = 440, = 240, = 150, = 88$ lbs. per sq. in.

The square column on its diagonal Tables C IV and C V

for $\frac{e}{a} = 0, = \cdot 1, = \cdot 3, = \cdot 5, = 1{\cdot}0$

has $\frac{L}{100} = 684, = 380, = 180, = 103, = 50$ lbs. per sq. in.

The circular column from Tables C VI and C VII

for $\frac{e}{D} = 0, = \cdot 089, = \cdot 267, = \cdot 444, = \cdot 89$

has $\frac{L}{100} = 684, = 395, = 195, = 112, = 60$ lbs. per sq. in.

The comparison is perhaps a little unfair to the column bent about a diagonal as the maximum stress occurs only at the extreme point. In practice the column would be stronger than theory indicates but, nevertheless, it is clearly weaker when flexed diagonally.

For all-round bending, therefore, a circular column is to be preferred although the difference is not great. Up to the present the square shape has been the favourite one in this country except for flat-slab work.

EXAMPLES ON CHAPTER VIII

A column is 12 ins. × 12 ins. with four ⅞-in. bars and $m = 18$. It carries a load of 50,000 lbs. (central) and a moment of 250,000 lbs.-ins. Find f_c and f_s.

$$p = \frac{2{\cdot}4}{12 \times 12} = 1{\cdot}67\%$$

$$e = \frac{250{,}000}{50{,}000} = 5'', \ \frac{e}{a} = \frac{5}{12} = 0{\cdot}417$$

From Table C III (*c*). $\dfrac{L}{abf_c} = 0{\cdot}39$ and $n = 0{\cdot}65$

$$\frac{50{,}000}{12 \times 12 \times f_c} = 0{\cdot}39 \text{ or } f_c = \textit{885 lbs. per sq. in.}$$

$$f_s = 885 \times 18 \times \frac{0{\cdot}25}{0{\cdot}65} = \textit{6100 lbs. per sq. in.}$$

A section 10 ins. × 20 ins. has 4 bars 1⅛ ins. It carries a load of 160,000 lbs. and a moment of 600,000 lbs.-ins. bending about the major axis. If $m = 12$ what is f_c?

$$p = \frac{4}{10 \times 20} = 2\%$$

$$e = \frac{600{,}000}{160{,}000} = 3{\cdot}75'' \text{ or } \frac{e}{a} = \frac{3{\cdot}75}{20} = 0{\cdot}187$$

From Table C II (*b*) $\dfrac{L}{abf_c} = 0{\cdot}625$

$$\frac{160{,}000}{10 \times 20 \times f_c} = 0{\cdot}625 \text{ or } f_c = \textit{1280 lbs. per sq. in.}$$

A section of an arch is 12 ins. wide by 16 ins. deep and has 3 bars 1⅛ ins. at top and bottom. It carries a thrust of 12,000 lbs. and a moment of 500,000 lbs.-ins. Find f_c and f_s if $m = 15$.

$$p = \frac{6}{12 \times 16} = 3{\cdot}12\%$$

$$e = \frac{500{,}000}{12{,}000} = 41{\cdot}7'' \quad \frac{e}{a} = \frac{41{\cdot}7}{16} = 2{\cdot}6$$

From Table C III (*a*) (continued)

$$\frac{L}{ab \times f_c} \times 600 = 56 \text{ and } n = 0{\cdot}41$$

$$\frac{12{,}000}{12 \times 16 \times f_c} \times 600 = 56 \text{ or } f_c = \textit{670 lbs per sq. in.}$$

$$f_s = 670 \times 15 \times \frac{0{\cdot}49}{0{\cdot}41} = 12{,}000 \textit{ lbs. per sq. in.}$$

A column is 20 ins. × 20 ins. with 8 bars 1⅛ in. arranged as Fig. 60. It carries a load of 150,000 lbs. and a moment of 1,200,000 lbs.-ins. What is f_c if $m = 15$?

The column has actually 8 sq. ins. of steel but as two bars are very near the neutral axis we will say that only 7 sq. ins. are effective.

$$\text{Thus } p = \frac{7}{20 \times 20} = 1{\cdot}75\%$$

$$e = \frac{1{,}200{,}000}{150{,}000} = 8'' \quad \frac{e}{a} = \frac{8}{20} = 0{\cdot}4$$

$$\text{From Table C III } (a). \quad \frac{L}{ab} \times \frac{600}{fc} = 225 \quad (n = \text{about } 0{\cdot}65)$$

$$\therefore \; f_c = \frac{600 \times 150{,}000}{20 \times 20 \times 225} = 1000 \text{ lbs. per sq. in.}$$

A circular section is 20 ins. diam. with 8 bars ⅞-in. The load is 90,000 lbs. and moment 415,000 lbs.-ins. What is f_c if $m = 15$?

$$p = \frac{100 \times 4{\cdot}80}{{\cdot}785 \times 20 \times 20} = 1{\cdot}53\%, \quad e = \frac{415{,}000}{90{,}000} = 4{\cdot}64''$$

$$\frac{e}{D} = {\cdot}232.$$

$$\text{From Table C VII, } \quad \frac{4\,L}{\pi\,D^2} \times \frac{600}{f_c} = 240.$$

$$\text{i.e., } f_c = \frac{600 \times 90{,}000}{{\cdot}785 \times 20 \times 20 \times 240} = 715 \text{ lbs. per sq. in.}$$

An octagonal column section is 18 ins. across the flats, it has 8 bars ¾-in. diam. and carries a load of 36,800 lbs. and a bending moment of 274,000 lbs.-ins. What is the maximum concrete stress if $m = 15$?

$$\text{Area of section} = {\cdot}828 \times 18^2 = 268 \text{ sq. ins.}$$

$$p = \frac{100 \times 3{\cdot}54}{268} = 1{\cdot}32\%, \quad e = \frac{274{,}000}{36{,}800} = 7{\cdot}45''.$$

Reduce the section to an equivalent circle of equal area we have

$${\cdot}785\,D^2 = 268 \quad \text{i.e.,} \quad D = 18{\cdot}5''$$

$$\frac{e}{D} = \frac{7{\cdot}45}{18{\cdot}5} = {\cdot}403.$$

$$\text{From Table C VII, } \quad \frac{4\,L}{\pi\,D^2} \times \frac{600}{f_c} = 135.$$

$$\therefore \; f_c = \frac{600 \times 36{,}800}{268 \times 135} = 610 \text{ lbs. per sq. in.}$$

A section of a chimney shell is 6 ins. thick and 6 ft. 6 ins. internal diameter. It carries a load of 150,000 lbs. and a bending moment of 1,120,000 lbs.-ft. If $m = 15$ what values of f_c and f_s are produced? The vertical steel in the chimney is 1% of the section.

Mean diameter $D = 7' \, 0'' = 84''$

$$e = \frac{1{,}120{,}000 \times 12}{150{,}000} = 89{\cdot}5''$$

$$\frac{e}{D} = \frac{89{\cdot}5}{84} = 1{\cdot}06.$$

From Table HC II we have $\dfrac{L}{D\,t\,f_c} = {\cdot}485$

$$\text{i.e., } f_c = \frac{150{,}000}{84 \times 6 \times {\cdot}485} = 612 \text{ lbs. per sq. in.}$$

Also $n = \text{about } {\cdot}164$ i.e., $\dfrac{f_s}{f_c} = \dfrac{{\cdot}664}{{\cdot}336} \times 15$

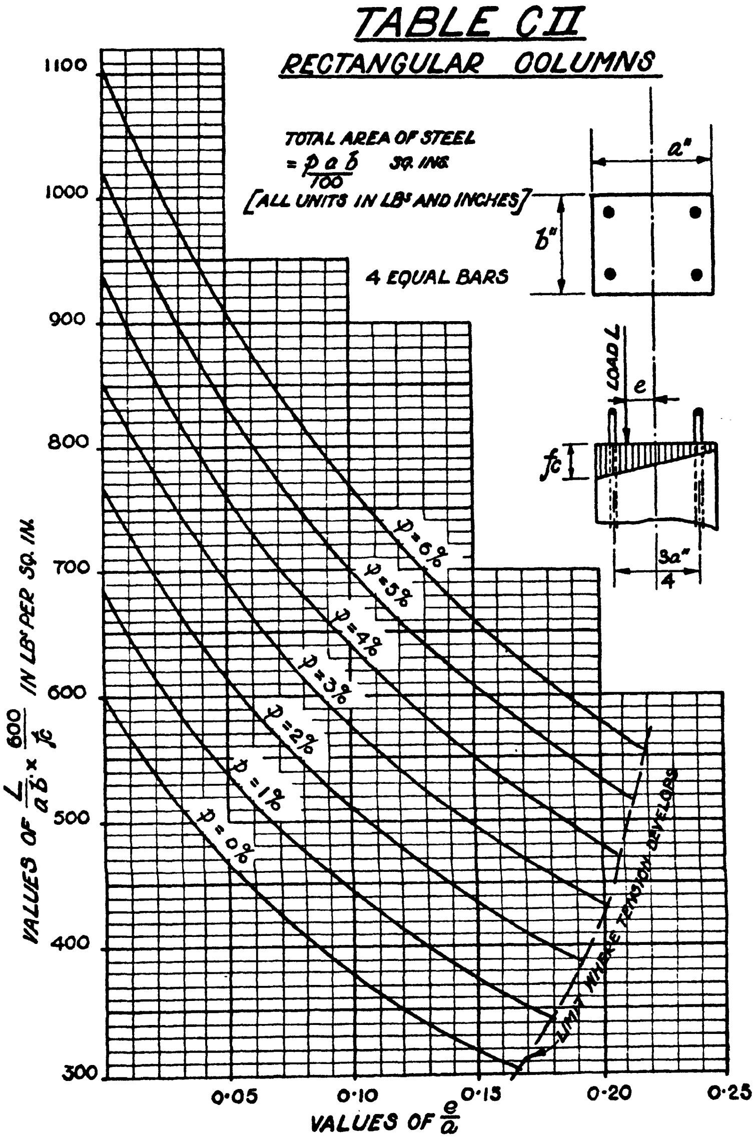
i.e., $f_s = 29{\cdot}6 f_c = 29{\cdot}6 \times 612 = 18{,}000$ lbs. per sq. in.

If we raise p to 2% we get $f_c = 480$ lbs. per sq. in.

$f_s = 11{,}000$ lbs. per sq. in.

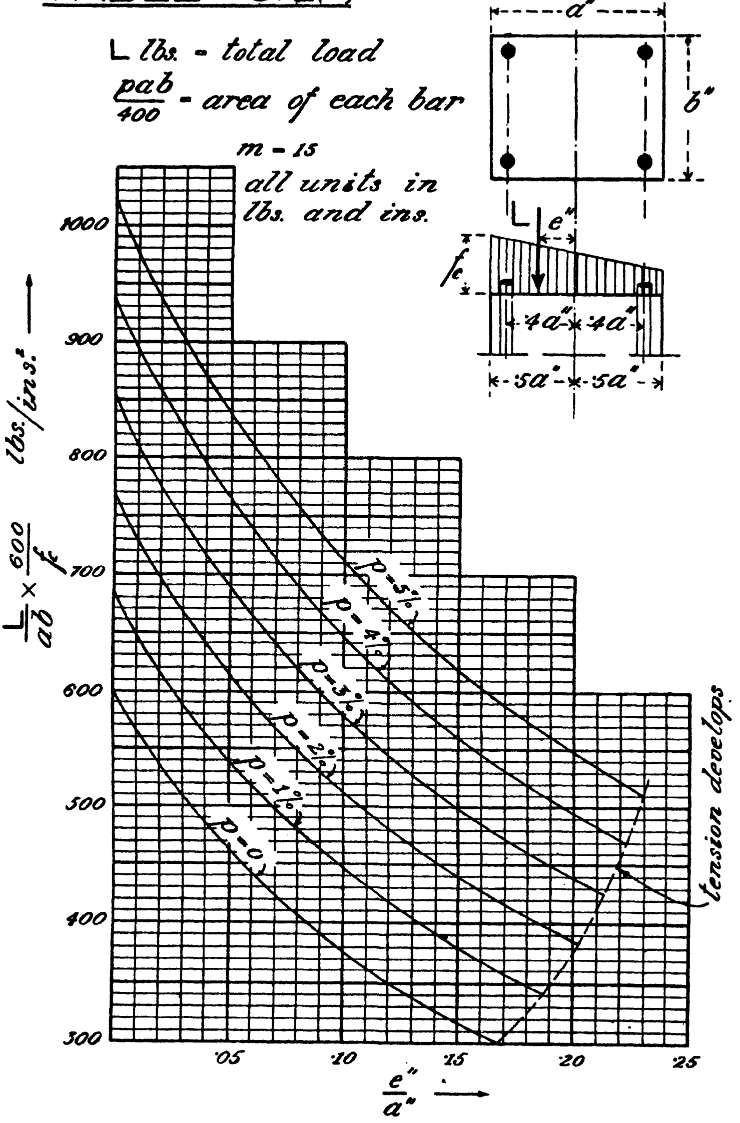
TABLE CII

RECTANGULAR COLUMNS



f_c = compressive stress on concrete (maximum).
$m = 15$.

TABLE C. II(a)



f_c = maximum compressive stress on concrete.

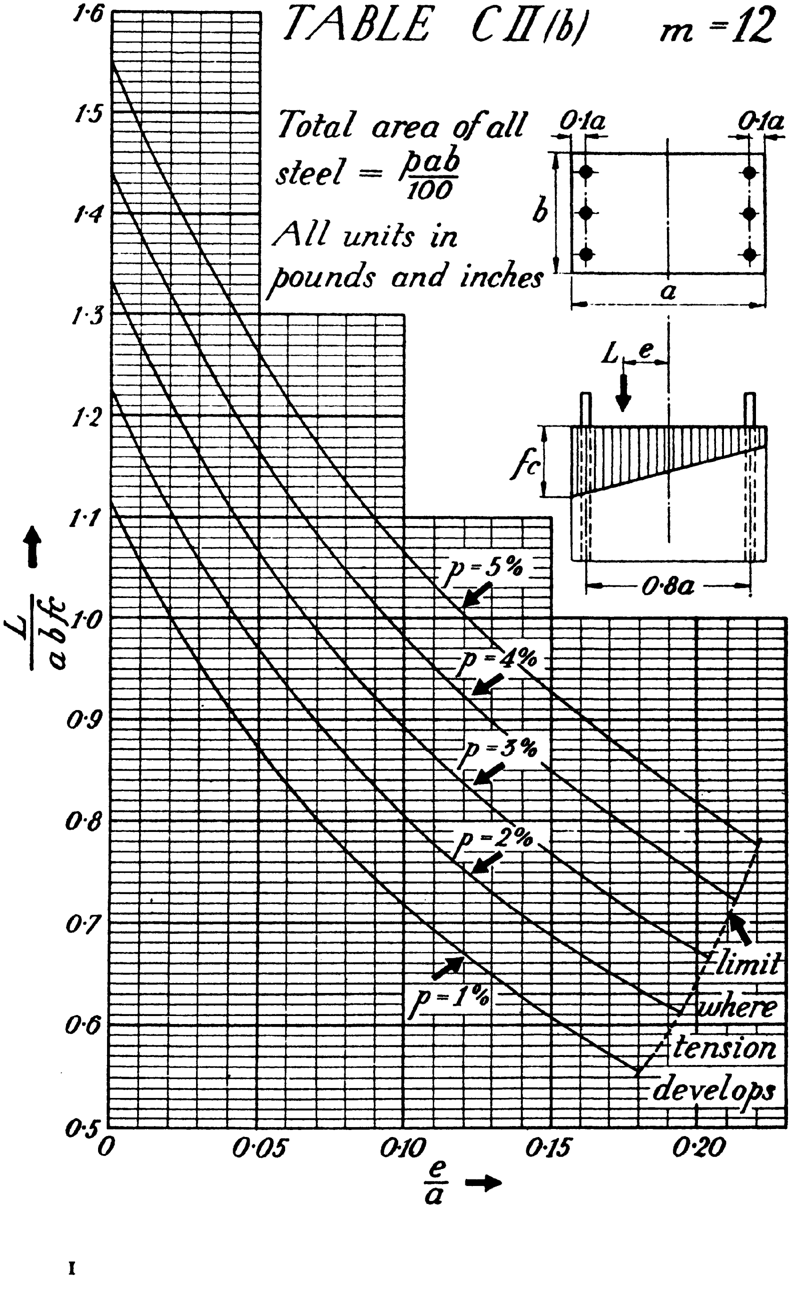


TABLE CII(b) m = 12
Total area of all steel = $\frac{pab}{100}$
All units in pounds and inches
0·1a
0·1a
b
a
L
e
fc
0·8a
$\frac{L}{abfc}$
1·6
1·5
1·4
1·3
1·2
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0
0·05
0·10
0·15
0·20
$\frac{e}{a}$
p = 5%
p = 4%
p = 3%
p = 2%
p = 1%
limit where tension develops

I

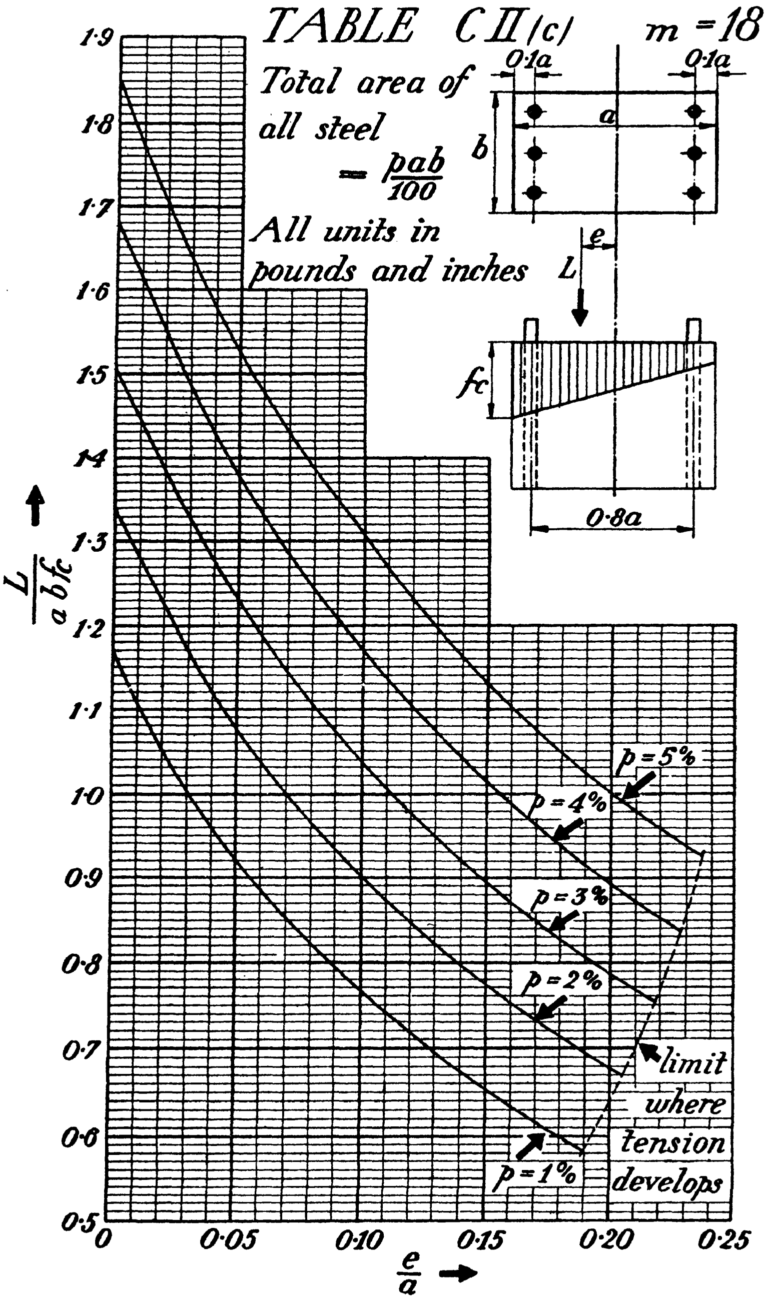


TABLE CII(c) m=18
Total area of all steel = pab/100
All units in pounds and inches
0·1a
0·1a
a
b
e
L
fc
0·8a
p=5%
p=4%
p=3%
p=2%
p=1%
limit where tension develops
L/abfc
1·9
1·8
1·7
1·6
1·5
1·4
1·3
1·2
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0
0·05
0·10
0·15
0·20
0·25
e/a

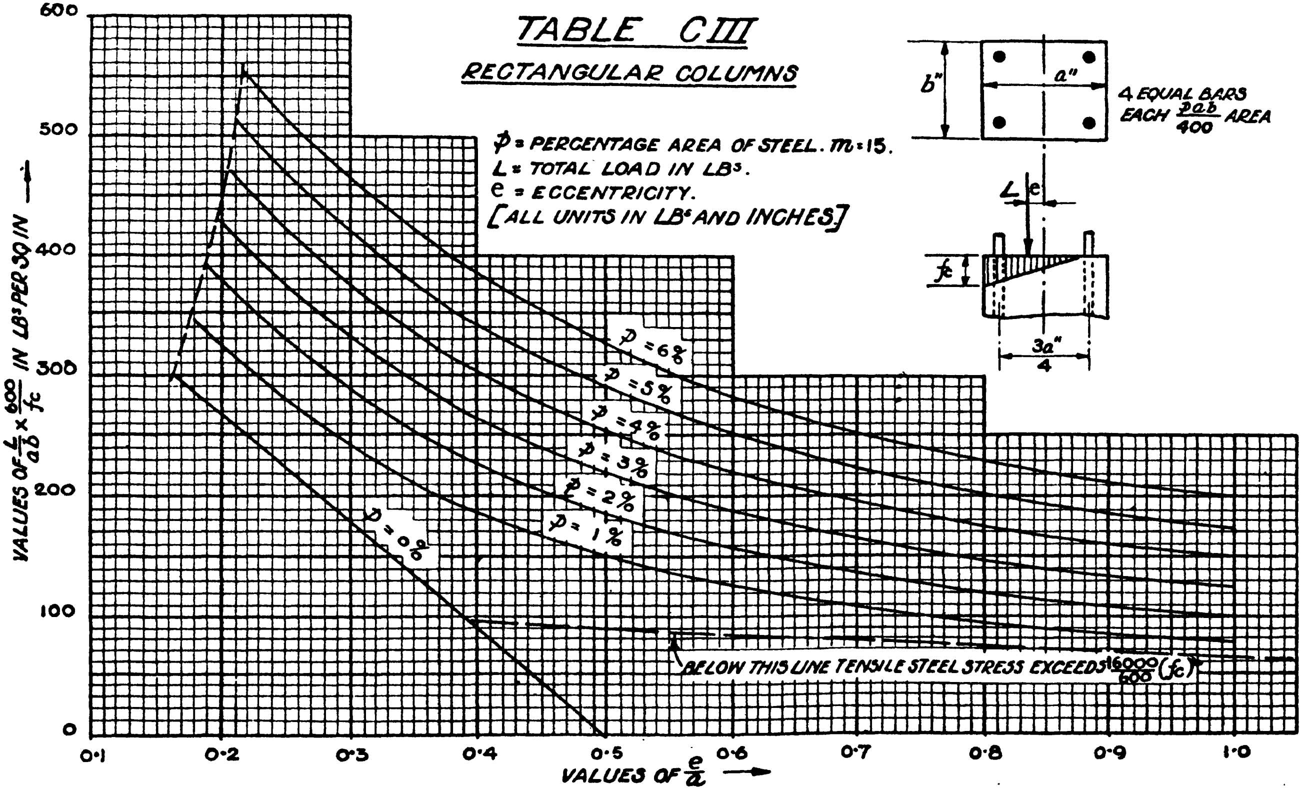
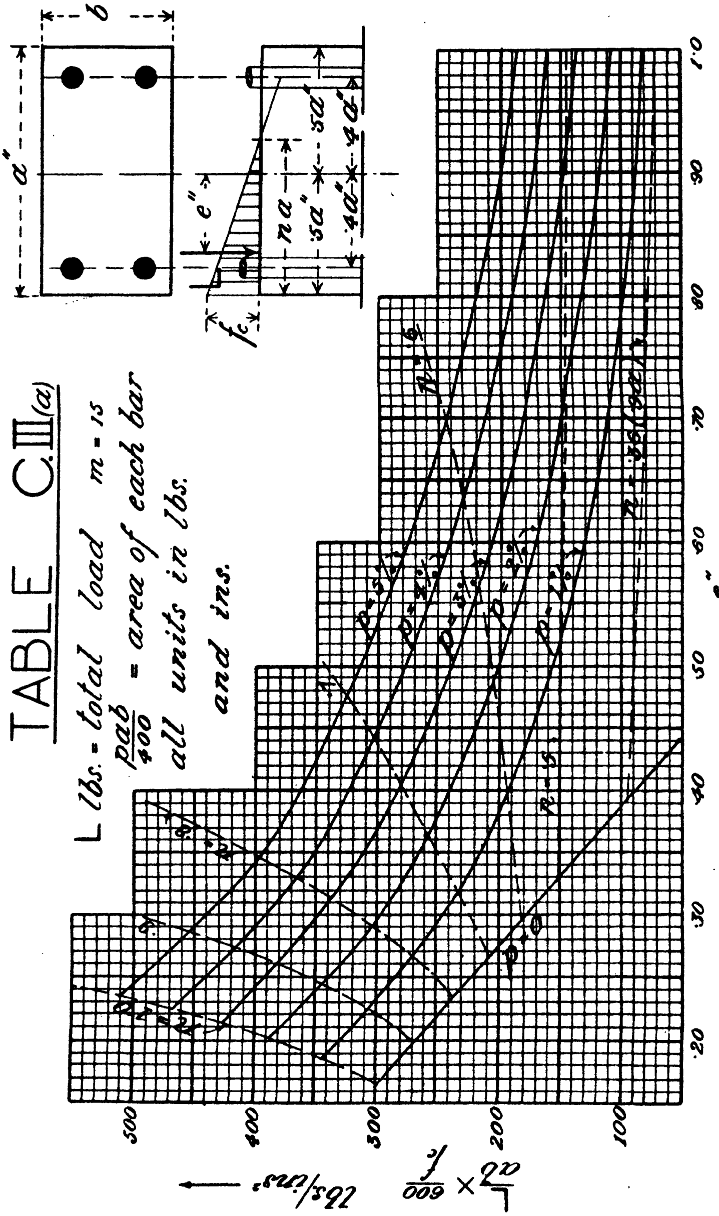


TABLE CIII

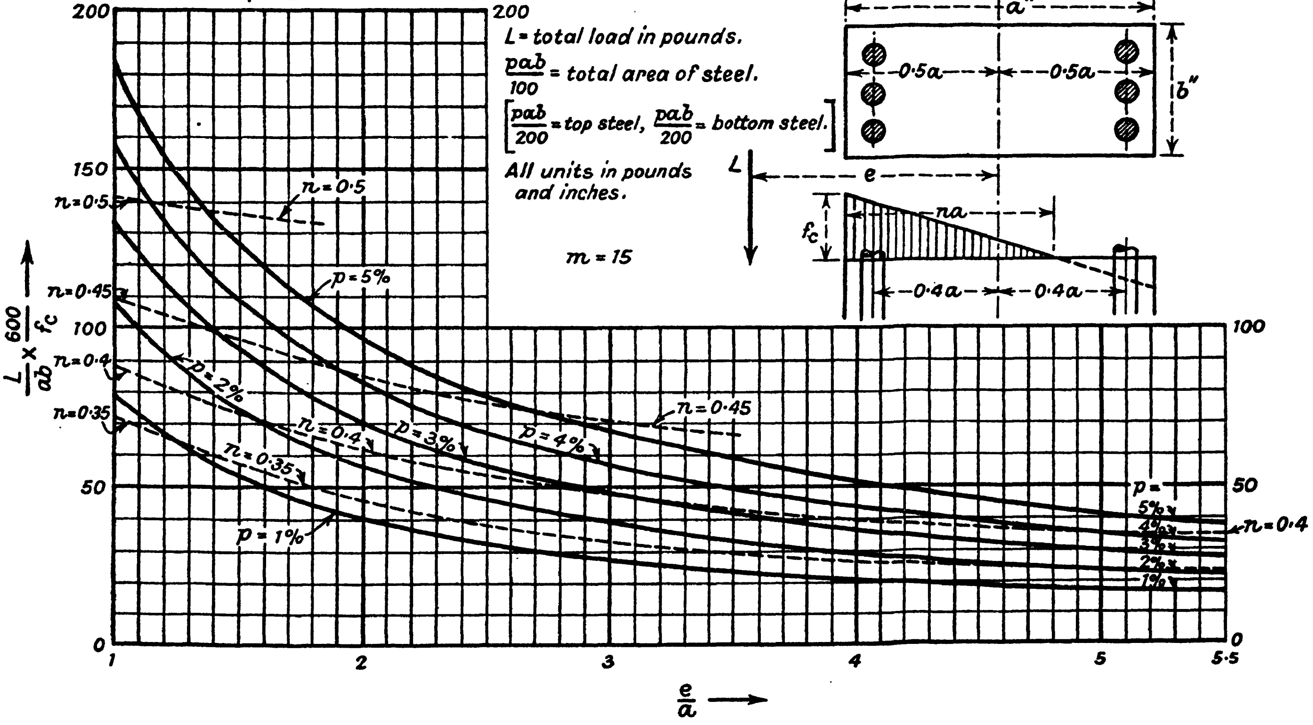
RECTANGULAR COLUMNS

p = PERCENTAGE AREA OF STEEL. m = 15.
L = TOTAL LOAD IN LBS.
e = ECCENTRICITY.
[ALL UNITS IN LBS AND INCHES]



f_c = maximum compressive stress on concrete.

TABLE CIII(a) (continued)



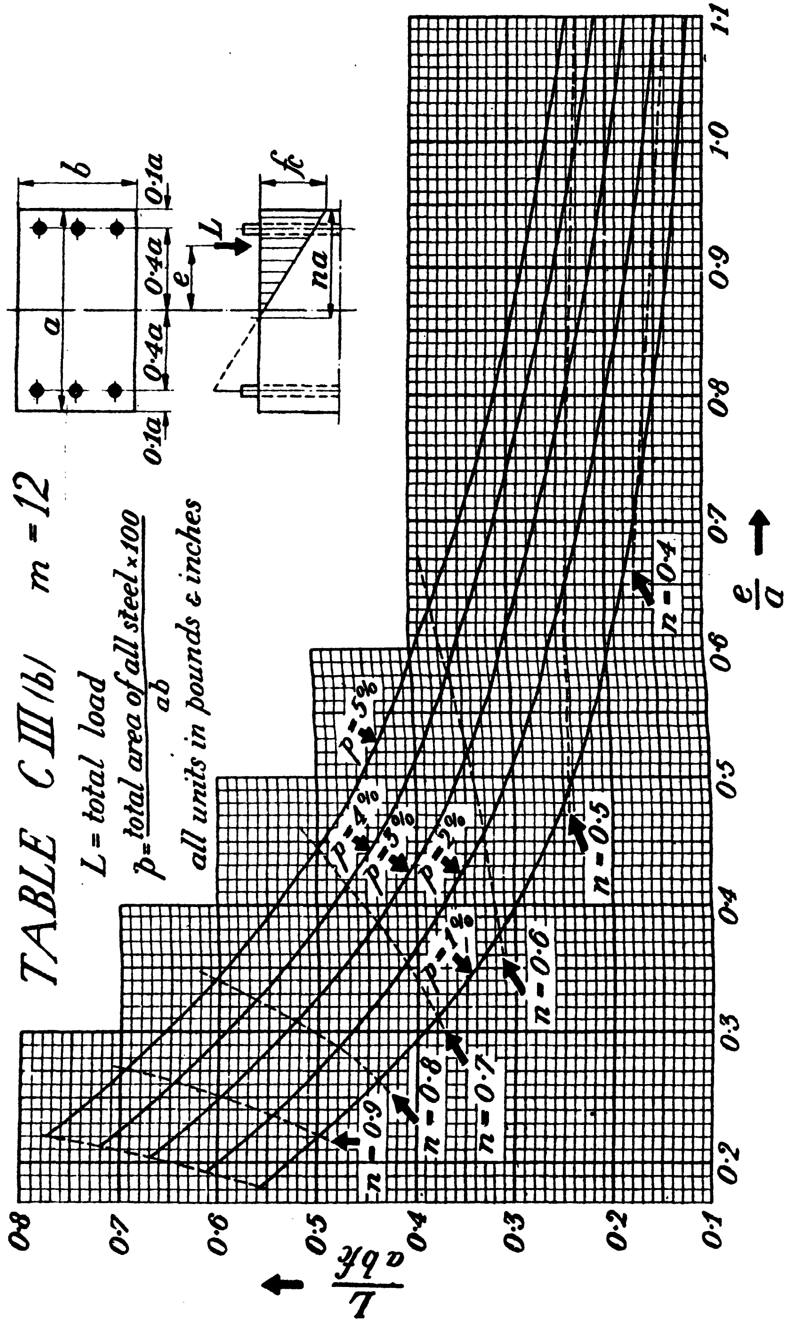


TABLE CIII(b) m = 12
L = total load
p = total area of all steel × 100 / ab
all units in pounds & inches
b
a
0·1a
0·4a
0·4a
0·1a
e
L
fc
na
p = 5%
p = 4%
p = 3%
p = 2%
p = 1%
n = 0·9
n = 0·8
n = 0·7
n = 0·6
n = 0·5
n = 0·4
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1·0
1·1
e/a
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
L/abfc

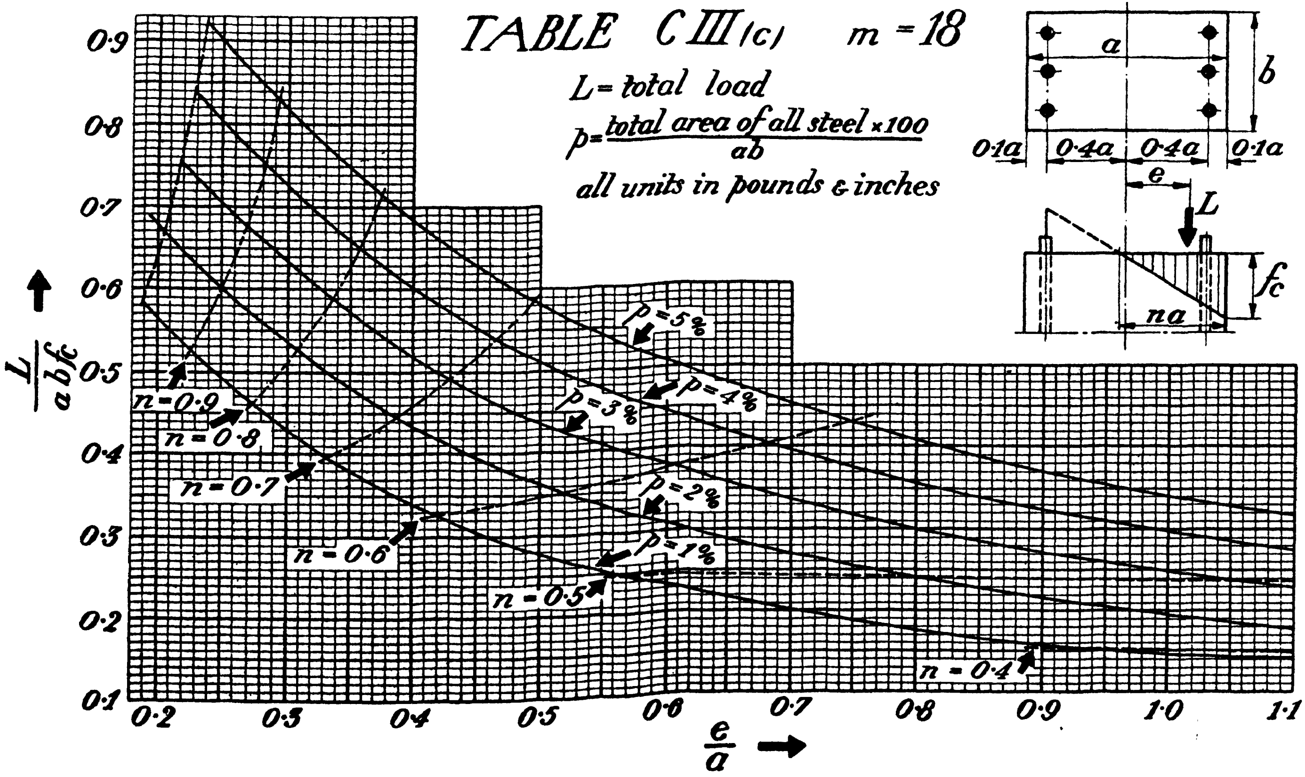


TABLE CIII(c) $m = 18$

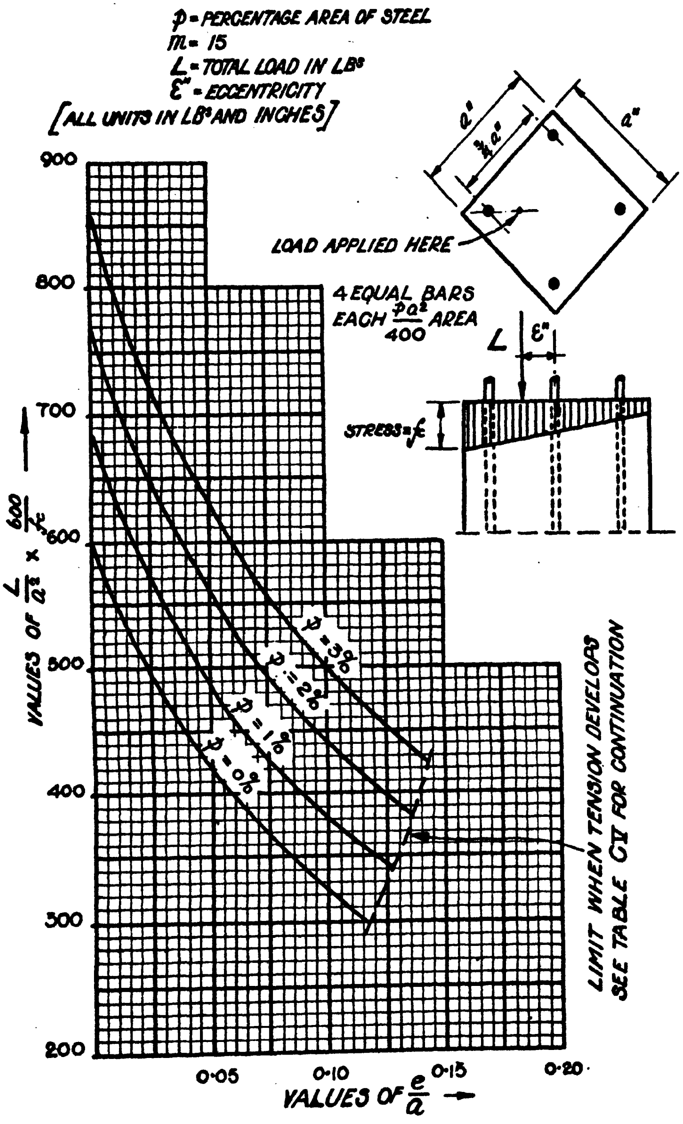
L = total load

$p = \frac{\text{total area of all steel} \times 100}{ab}$

all units in pounds & inches

—TABLE C IV—

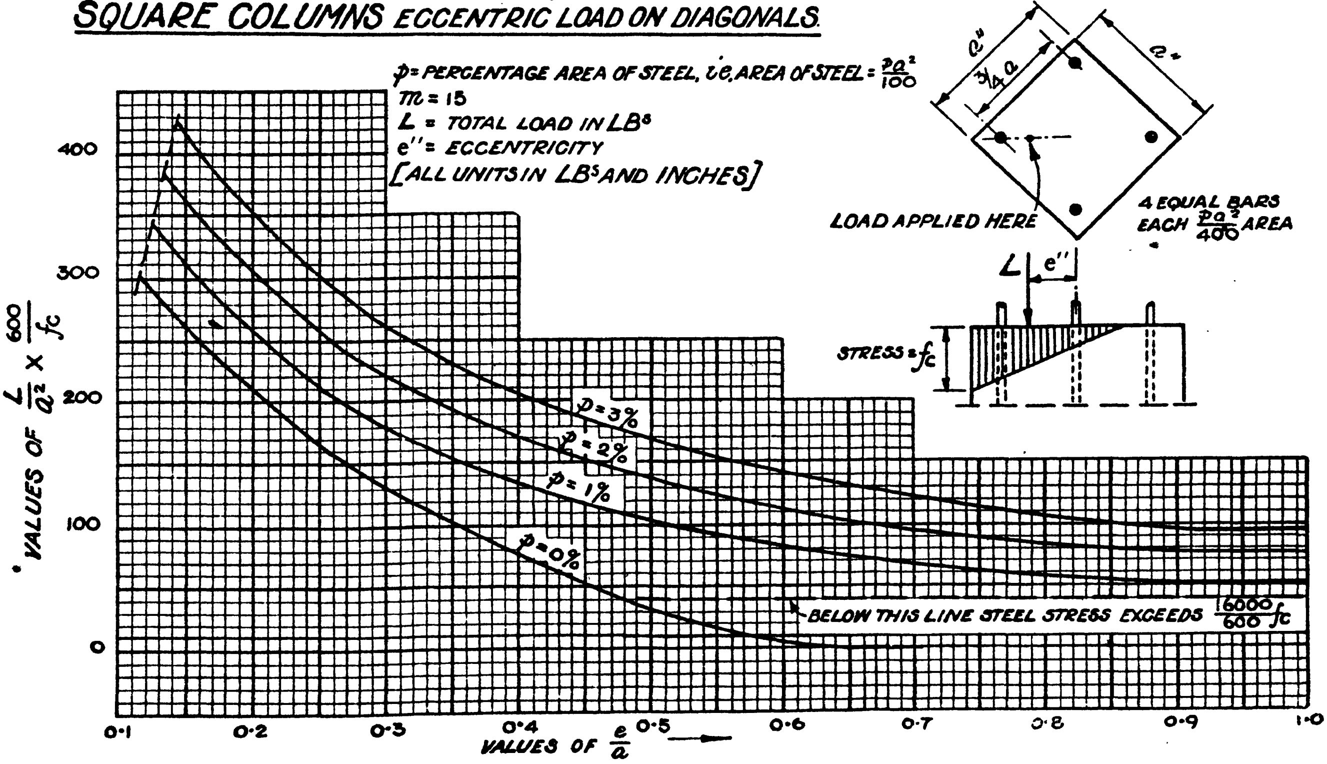
SQUARE COLUMNS, ECCENTRIC LOAD ON DIAGONALS.



The symbols e and ϵ both stand for the same value, namely the eccentricity.

TABLE CV

SQUARE COLUMNS ECCENTRIC LOAD ON DIAGONALS.



p = PERCENTAGE AREA OF STEEL, i.e. AREA OF STEEL = $\frac{pa^2}{100}$

$m = 15$

L = TOTAL LOAD IN LBS

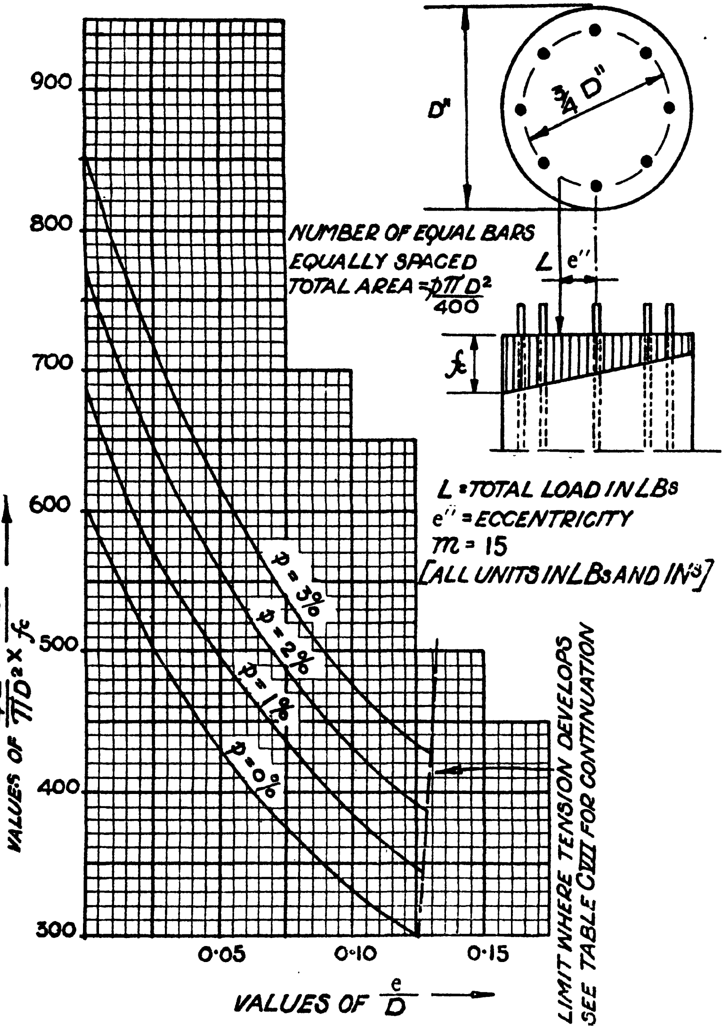
e'' = ECCENTRICITY

[ALL UNITS IN LBS AND INCHES]

TABLE CVII

CIRCULAR COLUMNS

USE ALSO FOR HEXAGONAL, OCTAGONAL, ETC



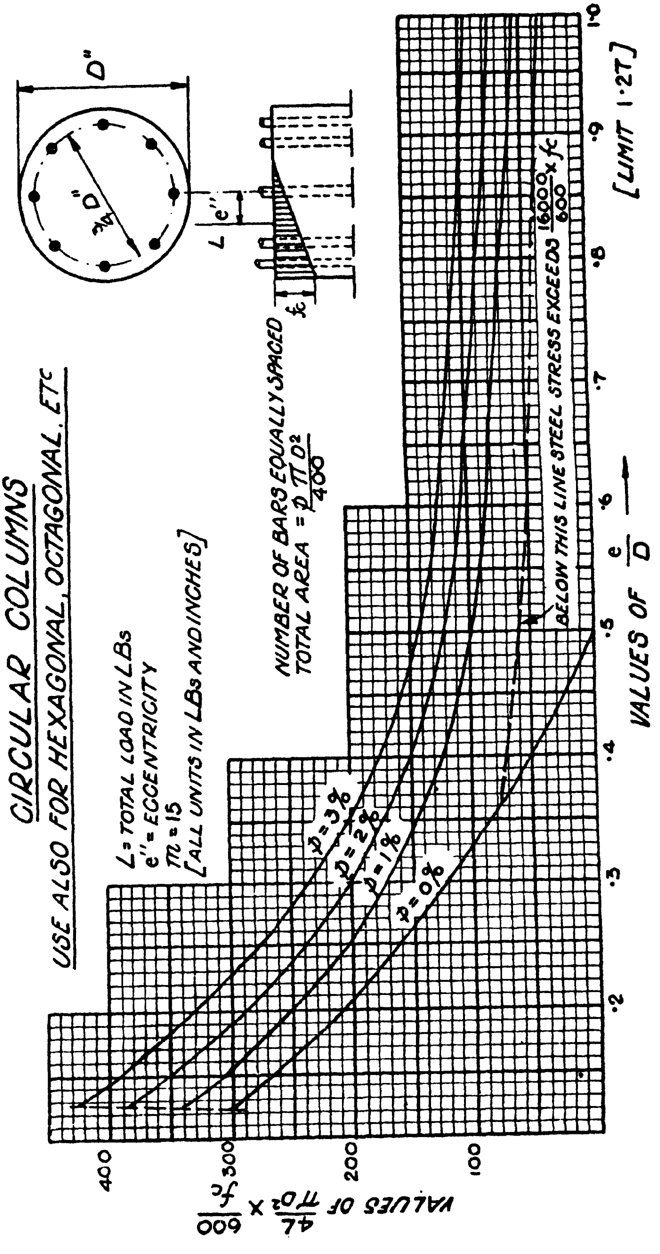


TABLE C VIII

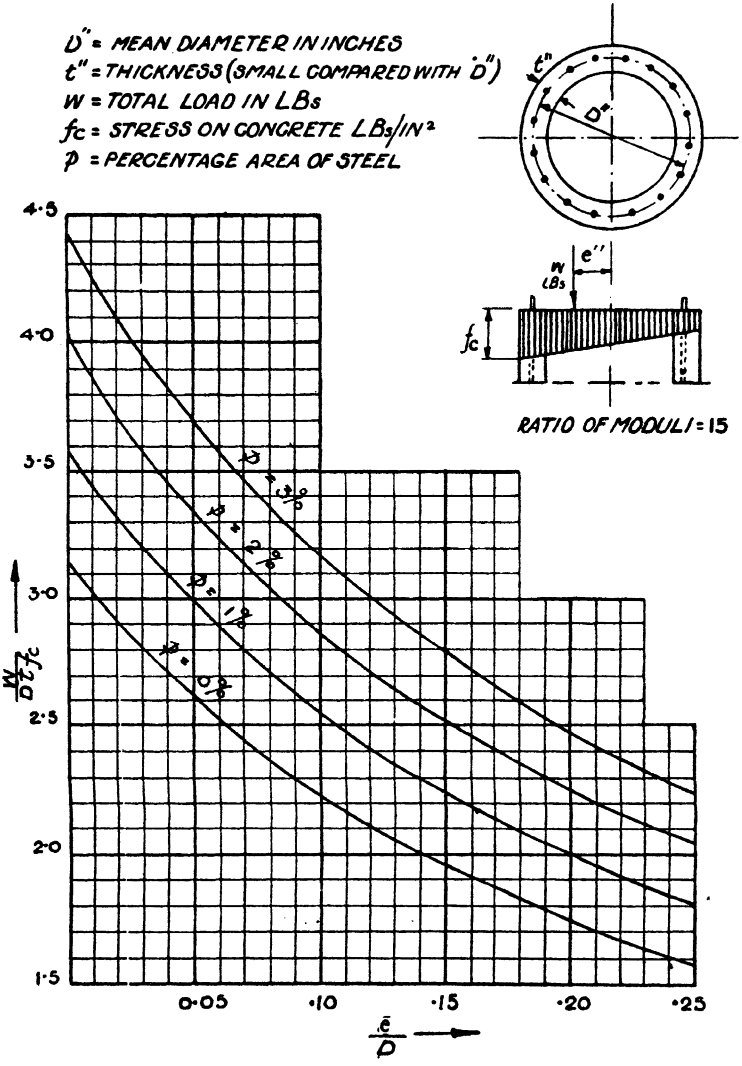
CIRCULAR COLUMNS

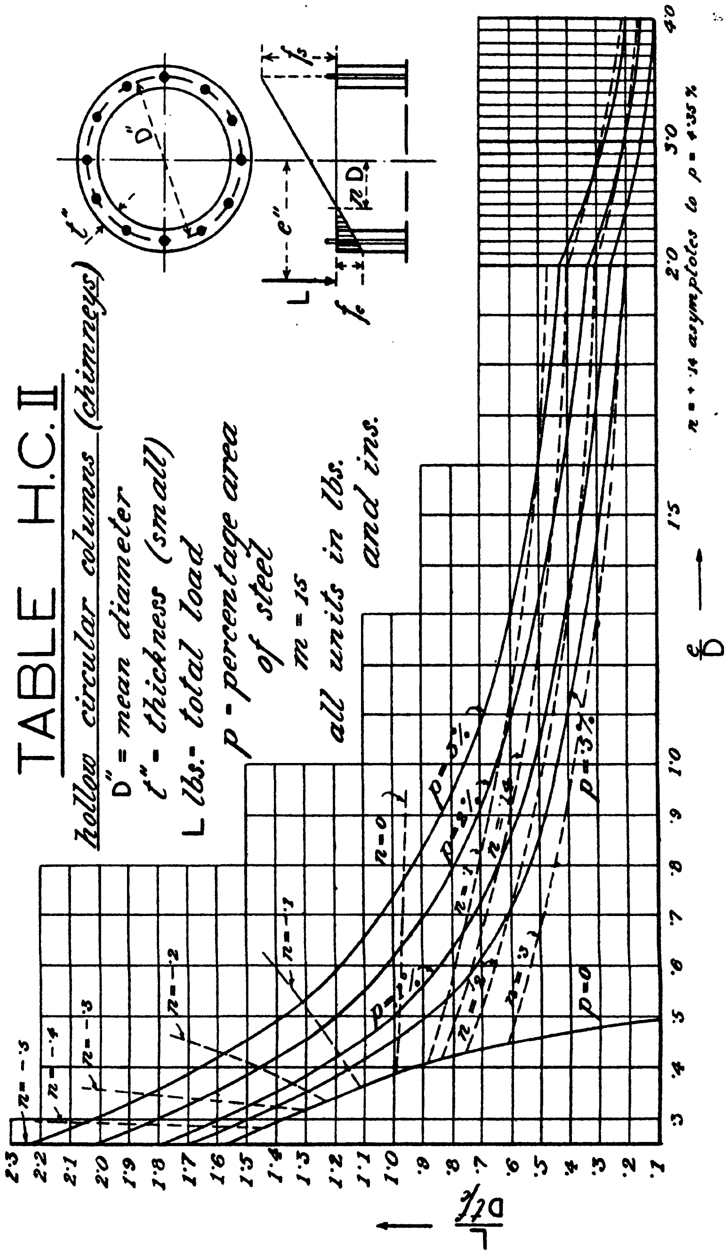
USE ALSO FOR HEXAGONAL, OCTAGONAL, ETC

TABLE H.C.I

HOLLOW CIRCULAR COLUMNS (CHIMNEYS)

D" = MEAN DIAMETER IN INCHES
t" = THICKNESS (SMALL COMPARED WITH D")
W = TOTAL LOAD IN LBs
fc = STRESS ON CONCRETE LBs/IN²
p = PERCENTAGE AREA OF STEEL





f_c = maximum compressive stress on concrete.

CHAPTER IX

SECTIONS UNDER SHEAR

The treatment of the problem in this chapter represents the usually accepted views on the subject. As will be seen it is inconsistent and rather cramped in outlook. A broader and more practical view is put forward in Chapter XXXV.

All the problems we have faced up to the present have been simplified by the fact that we have known the direction of the stresses all of which have been normal to the section.

The problem of shear is more complicated and demands not only a knowledge of the cross section of the member, but also a knowledge of the longitudinal section. It is, in fact, a two dimensional problem where the direction of the stresses on the concrete can only be approximately solved. By assuming different inclinations of the main stresses in the concrete, various theories of the strength of beams in shear can be evolved. This explains the many "new theories of shear" which appear from time to time in the literature of the subject.

The analysis which follows is based partly on a general view of the mathematical problem and partly on the results of tests. This analysis has been divided into four separate cases.

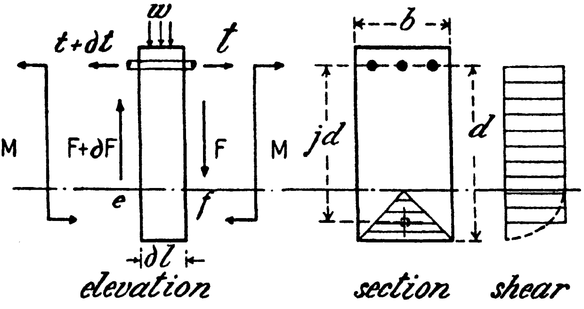


Fig. 62.

Case I. No Web Reinforcement.—Let Fig. 62 represent a case having no web reinforcement. The cross section of the beam is *b* ins. × *d* ins. effective depth. It has been assumed that tension occurs at the top and compression at the bottom edge as this is generally the case where the maximum shear occurs. The analysis, however, can readily be applied to all cases.

M = bending moment.

l = span of beam

F = total shear, w = load per unit length

t = total tension in tensile steel

jd = lever arm, i.e., the distance from the centre of the tensile steel to the centre of pressure of the compressed area.

Take a short length of beam $\delta\, l$.

Resolving all external forces vertically we have

$$F + \delta\, F = F + w \times \delta l$$

$$\text{i.e., } w\,\delta\, l = \delta\, F \quad \text{or} \quad w = \frac{\delta\, \mathrm{F}}{\delta\, l}$$

Taking moments for the external forces

$$M + \delta\, M = F\,\delta\, l + M$$

omitting small quantities of the second order

$$F\,\delta\, l = \delta\, M$$

$$F = \frac{\delta\, \mathrm{M}}{\delta\, l} \text{ a well-known result.}$$

This shows the equilibrium of the external forces applied to the short length of beam δl. We now come to the stresses set up by these forces.

We have

$$t = \text{Total tension on steel} = \frac{\text{Bending Moment}}{\text{Lever arm}}$$

$$t = \frac{M}{jd} \text{ and } \frac{M + \delta\, M}{j\, d} = t + \delta t$$

$$\delta\, t = \delta \left(\frac{M}{j\, d}\right) = \frac{\delta\, M}{j\, d} = \frac{F\,\delta\, l}{j\, d}.$$

Now suppose $e\,f$ is a section at the neutral axis.

The only stresses due to bending which occur above this line are those on the tensile steel for we have assumed that the concrete resists no tension. The shearing stresses on the two cross sections act vertically.

Therefore if we consider the equilibrium of that part of the short length of beam we are considering, which lies above $e\,f$ we have, by resolving horizontally

Total horizontal shear on $e\,f = (t + \delta t) - t = \delta t$.

The area of the section $e\,f = b\,\delta\, l$.

$\therefore$ Intensity of horizontal shear on $e\,f$

$$= \frac{dt}{b\delta l} = \frac{F\,\delta\, l}{j\, d} \times \frac{1}{b\,\delta\, l} = \frac{F}{b\, j\, d}.$$

Now if b is constant up the beam as drawn, the horizontal shear stress is constant from the tensile steel to the neutral axis. Beyond the neutral axis the compressive stresses in the concrete gradually reduce this shear to zero* as shown by the dotted line in the shear diagram on the right-hand side of Fig. 62. If, however, we assume that the compressive stresses are replaced by a single force acting at the centre of compression we can assume without much error that the shear stress is constant over the whole depth $j\,d$ as shown by the shaded area. We have therefore,

(a) The whole shear force is resisted by the " web area " $b\,j\,d$.

(b) The intensity of shear stress is constant over this area and $=\dfrac{F}{b\,j\,d}$.

The shear forces can be resolved by the well-known theorem that follows:

In Fig. 63 *abcd* is a small cube of material under pure shear, i.e., the faces *ab* and *cd* carry a shear force whose intensity $= f$ and no other stresses.

Now it is apparent that, in order to hold the cube in equilibrium, we must apply

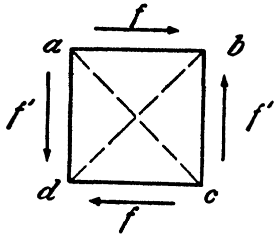


Fig. 63.

a stress tangential to the planes *ad* and *bc*.

If this stress $= f'$

Then by moments

$$f \times (ab)^2 \times bc = f' \times (bc)^2 \times ab$$
$$\text{i.e., } f = f'.$$

(The planes *ad* and *bc* may carry a normal stress in addition, but in the case under consideration we have assumed that the web area resists the whole shear and none of the bending stresses.)

The plane section *bd* carries a purely compressive stress. Consider the equilibrium of the half-cube *abd*.

Resolving normal to *bd*, we have

$$(f(ab)^2 + f'\,(ad)^2)\cos 45^\circ = \text{compressive stress} \times ab \times bd$$
$$\text{i.e., } \sqrt{2}fab^2 = \text{compressive stress} \times \sqrt{2}\,ab^2$$

$\therefore$ compressive stress on $ab = f$.

Similarly the plane *ac* carries a purely tensile stress whose intensity $= f$.

---

* See Morley's *Strength of Materials*, for a fuller discussion.

If the cube *abcd* were made of plain concrete it would tend to fail by tension across the plane *ac*.

Tests and general practice show that a beam having no web reinforcement will safely carry a shear stress, calculated as above, of 75 lbs. per sq. inch, if made of average 1 : 2 : 4 concrete.

If the shear intensity exceeds this figure then the web requires reinforcement to take the inclined tension. .

In most floor and wall slabs this figure of 75 lbs. per sq. in. is not exceeded. On the other hand nearly all T-beams show a calculated shear intensity of two to four times this amount.

Beams with Web Reinforcement. Direction of Compressive stress in Concrete.—We have shown that we may take the shear stress on the web of a beam as being uniformly distributed over the " web area " $b j d$. We shall assume that the web compression in the concrete is uniform over the same area.

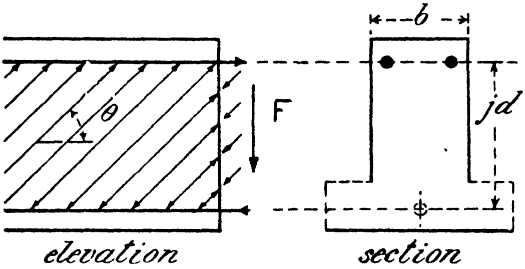


Fig. 64.

Let this compressive stress be s_c and the angle that it makes with the neutral axis be θ. Then at any section such as shown in Fig. 64 the vertical component of this concrete stress on the cross section $b j d$

$$= s_c \times b j d \cos \theta \sin \theta$$

$$= b j d s_c \frac{\sin 2 \theta}{2}$$

This is obviously a maximum when $2 \theta = \frac{\pi}{2}$

i.e., when $\theta = 45°$.

It is possible to assume any angle we choose for θ since we are assuming that the concrete is not strong enough to resist the web tension and, having cracked under the stress, has therefore ceased to function as a homogeneous elastic solid. Some authors have assumed that θ varies with the type of web reinforcement employed. From the principle of least work however it appears that an angle of 45° is the most probable value of θ and this agrees with the calculated value at the neutral axis for a homogeneous solid.

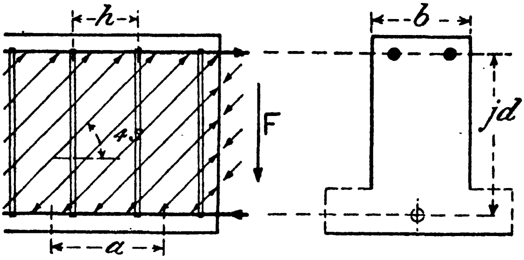


Fig. 65.

Case II. Web Reinforced with a vertical stirrup-system.—Fig. 65 represents a case with vertical stirrups only. The concrete stress consists of a compression s_c at 45° to the neutral axis.

We can imagine that the concrete in the web consists of a row of small independent columns inclined at 45° and that the ends of these columns are supported on the tension and compression flanges.

The horizontal components of these end reactions are of course taken up by the flange stresses which increase or decrease at a rate in direct proportion to the shear as already shown since $F = \frac{\delta M}{\delta l}$. The vertical components are taken by the vertical stirrups.

Let s_c = compressive stress in concrete.

Since we have vertical bars only, there are no bars crossing a vertical section.

∴ vertical component of concrete stress at a vertical section

$$= \frac{s_c \times j\,d\,b}{2} \text{ as in the previous analysis}$$

$$= \text{total vertical shear}$$

$$= F$$

or $$s_c = \frac{2\,F}{j\,b\,d}.$$

Now take a short length of beam a.

Total inclined compression on the area $a \times b = \frac{ab\,s_c}{\sqrt{2}}$.

Resolving vertically.

$$\text{Total vertical stress} = \frac{ab\,s_c}{2} = \frac{ab \times 2\,F}{2\,j\,d\,b} = \frac{a\,F}{j\,d}$$

$$\text{vertical tension per unit length of beam} = \frac{F}{j\,d}.$$

If s_s = safe tension stress in stirrups
i = effective area of each stirrup in tension
h = horizontal spacing of stirrups at the point where the total shear force = F.

Safe tension on stirrups per unit length

$$= \frac{s_s \times i}{h} = \frac{F}{j\,d}, \quad \frac{i}{h} = \frac{F}{j\,d\,s_s} \quad \text{or } F = \frac{j\,d \times s_s i}{h}$$

i.e., Sectional area of stirrups per unit length

$$= \frac{\text{Shear}}{\text{lever arm} \times \text{safe stress}}.$$

Care should be taken to make the units the same, as the areas of steel in the bar-tables are sq. ins. per *foot* run of section while $j\,d$ will be in inches.

We have

$$\text{sectional area of stirrups per foot run} = \frac{12 \times \text{total shear}}{(\text{lever arm}) \times (\text{safe stress})}$$

It might appear that vertical stirrups would not be so effective as stirrups inclined at 45° but it must be remembered that the main bars in the tension flange not only function in their own immediate vicinity, but also, owing to the strength of the concrete and its adhesion to the bars, prevent vertical cracks opening on the tension side of the neutral axis. In specimens of beams tested to failure the shear cracks as a rule show a very small inclination to the neutral axis. Vertical stirrups owing to the ease with which they can be anchored round the bars and the small interference they offer to proper concreting provide the more satisfactory form of web reinforcement.

Case III. Web Reinforced with an inclined stirrup system.—This case seldom arises except in very large beams having a large number of main bars which can be bent up close enough to one another to form a continuous system.

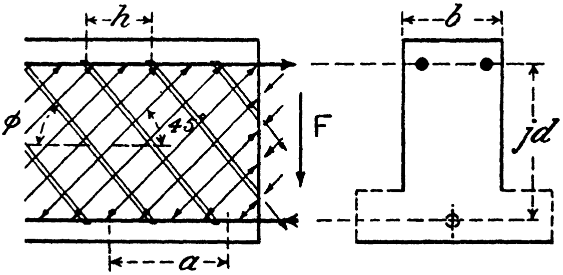


Fig. 66.

Suppose we have a system of stirrups inclined at $\Phi°$ to the neutral axis and spaced at a horizontal distance of h apart as in Fig. 66.

Let the tensile stress in the stirrups be s_s and i the area in tension of each stirrup.

Let s_c be the compressive stress in the concrete which we shall take as being inclined at 45° as in the previous cases.

Take a short length a.

The total pressure on the area ab due to the compression in the concrete

$$= ab \times \frac{s_c}{\sqrt{2}}$$

due to the tension in the stirrups

$$= \frac{s_s i}{h} \times a.$$

The horizontal components of these forces are taken up by the flange stress.

The vertical components must be equal

$$\therefore \quad \frac{ab \times s_c}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{s_s i}{h} \times a \sin \Phi$$

$$\text{or} \qquad s_c = \frac{2\,i}{hb} \times s_s \sin \Phi.$$

Taking a vertical section of the beam web, which is $j\,d$ deep, number of stirrups cut by section $= \dfrac{j\,d}{h \tan \Phi}$.

Total inclined stress on stirrups cut by vertical section

$$= s_s i \times \frac{j\,d}{h \tan \Phi}.$$

Vertical component of stirrup tension

$$= \frac{s_s i \times j\,d}{h \tan \Phi} \sin \Phi = \frac{s_s i \times j\,d}{h} \times \cos \Phi.$$

Vertical component of inclined compression in concrete

$$= \frac{j\,d\,.\,s_c \times b}{\sqrt{2} \times \sqrt{2}}.$$

$$\therefore \quad \text{Total shear} = F = \frac{s_c j\,d\,b}{2} + \frac{s_s i \times jd}{h} \cos \Phi$$

putting in the value of s_c obtained above

$$F = \frac{2\,i}{h\,b} s_s \sin \Phi \times \frac{j\,d\,b}{2} + \frac{s_s i \times j\,d}{h} \times \cos \Phi$$

$$= \frac{j\,d \times s_s i}{h} [\sin \Phi + \cos \Phi]$$

$$= \frac{\text{strength of each stirrup} \times \text{lever arm}}{\text{(horizontal spacing)}} \times [\sin \Phi + \cos \Phi].$$

If we write $\Phi = 90°$ i.e., when the stirrups are vertical we get

$F = \dfrac{j\,d\,s_s i}{h}$ as in the analysis of this special case.

If we write $\Phi = 45°$

we have $F = \frac{j\,d\,s_s\,i}{h} \times \sqrt{2}$

i.e., a stirrup system at 45° is $\sqrt{2}$ times as effective as one at 90° with the same horizontal spacing. On the other hand the length of each stirrup would be increased $\sqrt{2}$ times so the efficiency per weight of steel is the same. As the vertical system is easier to anchor and place it is generally preferred.

Case IV. Inclined Bar (single Bar or Intermittent System).—In the two cases (Case II and Case III) that we have already analysed we have assumed that the ends of the small inclined columns which carry the compression in the concrete are continuously supported on the tension and compression flanges.

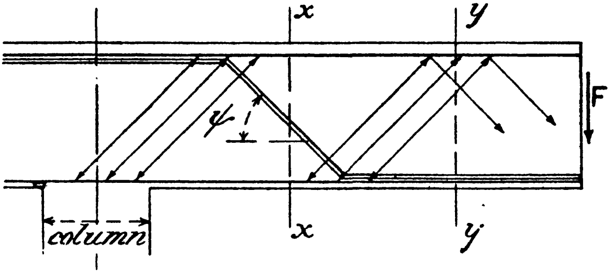


Fig. 67.

If the stirrups are closely spaced this is a very fair assumption. If, however, we have only one or two bent up bars there will be places where no support is available.

Thus in Fig. 67 which shows a beam having one bent up bar, only the ends of those columns which occur at or near the bends in the bar can be taken as being adequately supported (together with those that rest directly on the column or other support).

Beyond some such section as $y - y$ the tensile stresses in the web must be provided for. It may so happen that the shear force, which is almost invariably a maximum at the column, diminishes to such an extent that the concrete alone may be sufficient to take all the shear beyond $y - y$. If not, then another bent bar or a system of stirrups must be provided.

Taking a vertical section $x - x$. Since no stress is taken by the concrete we have

Shear = Total stress in bar × sin ψ.

Application. Slabs.—Floor slabs and wall slabs are invariably constructed without stirrups and the following shear stresses may be worked to without any further calculation.

(a) Slabs with no bent-up bars, 75 × $b\,j\,d$ lbs.

(b) Slabs where half the tensile steel is bent up and half taken through, 100 × $b\,j\,d$ lbs.

Beams. Limiting Total Shear Stress.—We could, theoretically, by increasing the amount of web reinforcement, increase the shear strength of the beam until the inclined compression in the concrete reached its maximum safe value. If the concrete will safely resist 750 lbs. per sq. in. in bending, we might expect that we could, by providing sufficient steel to take the tension, increase the safe shear strength of the beam up to 750 lbs. per sq. in. on the $b\,j\,d$ area. In practice the provision of so much steel makes it impossible to place the concrete properly and instead of a sound, solid beam we have the concrete full of air pockets and cut up into small disjointed pieces by closely spaced rows of bars.

In any case it is impossible to utilize fully the concrete outside the outermost stirrups and in all cases we get heavy local stresses and high bond stresses.

It is clear that there is a limiting point where increasing the amount of shear steel begins to decrease the total shear strength of the beam. This point will occur much sooner in practical contracting than in beams made under laboratory conditions.

Beams which are very deep compared with their span are stronger than shallow beams.

If the safe working stress on the concrete in bending is x lbs. per sq. in., then the total shear should not exceed the following on the $b\,j\,d$ area.

For beams whose depth is one-sixth of the span or less
$0{\cdot}3\,x$ but not exceeding 300 lbs. per sq. in.

For beams whose depth is one-sixth to one-third of the span
$0{\cdot}4\,x$ but not exceeding 400 lbs. per sq. in.

For beams whose depth is more than one-third of the span
$0{\cdot}5\,x$ but not exceeding 500 lbs. per sq. in.

The author is aware that a maximum total shear of 600 lbs. per sq. in. on the $b\,j\,d$ area has been suggested for shallow beams, but in the course of his experience he has never seen any design of web steel nor any arrangement of steel on site, nor any concrete sufficiently well made and placed to justify such a high working stress.

Total Effective Shear Strength of Beam.—We have discussed three methods of carrying shear stresses in a beam.

(1) By diagonal tension in the concrete plus diagonal compression in the concrete.

(2) By tension in a system of steel stirrups (vertical or inclined) plus diagonal compression in the concrete.

(3) By tension in bent-up bars plus diagonal compression in the concrete.

If we keep below the values given above for the limiting total shear, the total diagonal compression in the concrete will be quite safe.

If the value of $\frac{F}{b\,j\,d}$ does not exceed the safe shear stress on plain concrete (about 75 lbs. per sq. in. on average 1 : 2 : 4 mix) we need supply no shear reinforcement. If the value $\frac{F}{b\,j\,d}$ exceeds the safe shear stress on plain concrete we may assume as a working hypothesis that the concrete will crack and all the shear tension must be taken either by stirrups or by bent-up bars or by a combination of both. The total shear strength of the beam, therefore, if $\frac{F}{b\,j\,d}$ exceeds the safe shear stress on plain concrete, is the shear strength of the stirrup system plus the shear strength of the bent-up bars (provided always that this total does not exceed the limiting total shear stress given above).

It would appear (in theory) that, if an inclined bar bent up at 45° (say) is fully stressed in tension then a vertical stirrup in the same vicinity cannot be fully stressed as the two cases presuppose different strains in the material. The reader might imagine that method (2) (stirrups) and method (3) (bent-up bars) cannot both be fully operative at the same time. In practice, however, it is quite safe to work out the strength of each separately and add them together.

Working Stresses in Shear Reinforcement.—For stirrups which are anchored round main bars both ends, and bent-up bars having at least 45 diameters length beyond each bend 18,000 lbs. per sq. in.

For stirrups anchored one end only 12,000 lbs. per sq. in.

The above are fair working stresses based on the size and grip length of the bars that would normally be employed.

Spacing of Vertical Stirrups.—Vertical stirrups spaced at a horizontal distance equal to or less than half the effective depth of the beam can be taken as a continuous stirrup system. Vertical stirrups spaced at a horizontal distance equal to the effective depth are useless to resist shear as shear cracks are found to develop between them. They are, however, useful as ties and are invariably inserted at points where no stirrup system is necessary. For spacings varying from $\frac{1}{2}\,d$ to d, the stirrup system adds a little to the shear strength of the beam and the designer can interpolate his own values should such cases ever occur.

Spacing of Bent Bars.—In order to be regarded as a continuous system the bars must be spaced horizontally at a distance equal to or less than d and Φ must be equal to or less than 45°. If spaced farther apart than this they should be treated as " single bars". " Single bars " must be spaced not farther apart than about twice the depth of the beam or there will be planes of weakness between the bottom bend of one bar and the top bend of the next.

Bars bent up at smaller angles than 45° may be spaced somewhat farther apart, roughly up to a distance of $(d + d\cot\Phi)$ as in case (3) in Fig. 68.

Thus in Fig. 68 example No. (1) may be taken as a continuous system from $a - a$ to $b - b$ and as a single bar from $b - b$ to $c - c$.

Beyond *c* — *c*, if the concrete were not able to take the whole shear, stirrups or further bars would be necessary. Examples No. (2) and No. (3) could both be taken as single bars up to the points *e* — *e* and *h* — *h* respectively. In example (4) however, the distance between *j* — *j* and *k* — *k* is excessive and the bars must be brought closer or stirrups introduced to take the whole of the excess shear (beyond what the concrete will safely carry) between these two sections.

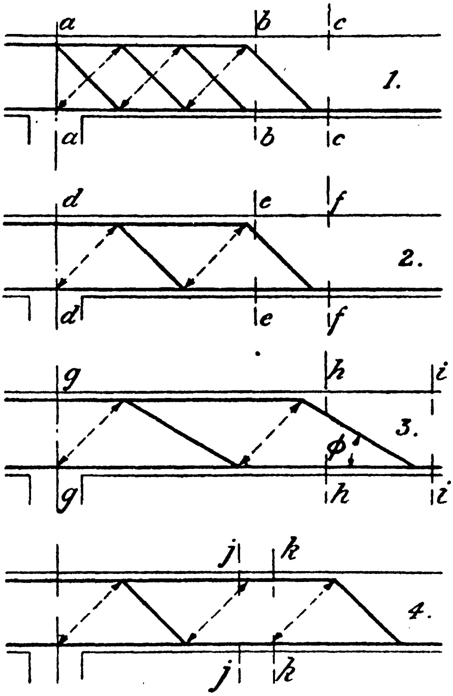


Fig. 68.

Occurrence of the Problem.—In practice the main bars and section of the beam are usually determined by considerations of the bending moment at the mid-span section.

We are therefore already given the concrete area and the bars available for bending up. We have then to design the stirrups to take the remainder of the shear tension (if any). If the concrete, in conjunction with the bent-up bars, is sufficient to carry the whole of the shear, then small stirrups spaced at a distance equal to the depth of the beam or somewhat less will be provided to act as ties near the centre of the span. Similar stirrups spaced at a distance of about half the depth of the beam would be provided near the supports in a continuous beam to tie in the compression steel (compare links in columns).

In beams carrying heavy rolling loads where the centre portion of the beam may have to carry heavy shears in either direction a vertical stirrup system will obviously be adopted as such a system can carry shear of either kind, whereas bent-up bars are only effective in one direction. The subject will be discussed further when we are in a position to work out the actual sections of the beams and the shears which they have to carry.

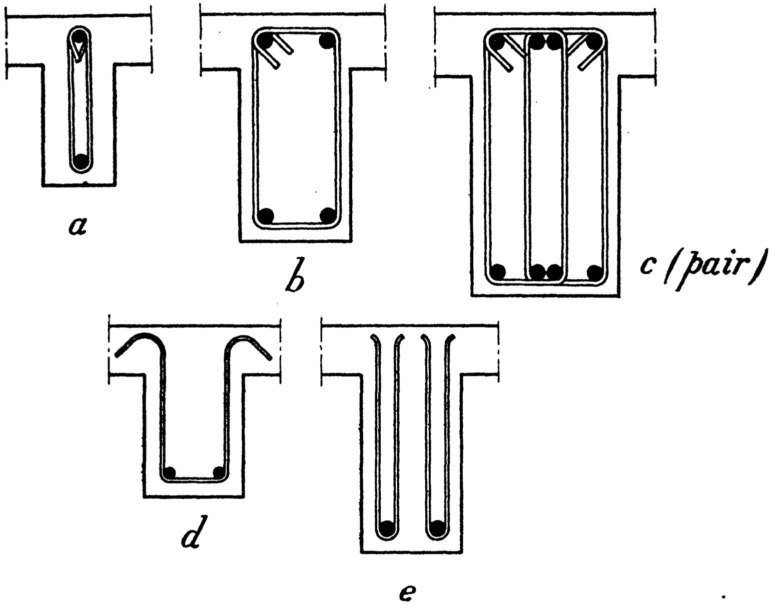


Fig. 69.

Usual shapes of stirrups are shown in Fig. 69.

They should whenever possible be anchored round top bars and as a rule such bars are specially provided where no top steel would otherwise occur.

Failing this the ends may be bent over as shown in (*d*). It should be noted that all the above stirrups have two legs and that the area in tension per stirrup is twice the area of the bar.

Example.—A T-beam is 8 ins. × 18 ins. overall, the flange thickness being 4 ins. At one section it has two bars 1 in. diameter bent up side by side at 30° to the horizontal (well anchored) and stirrups as (*b*) in Fig. 69 $\frac{3}{8}$-in. diam. at 4 ins. centres. The safe shear on plain concrete is 75 lbs. per sq. in. What total shear will the beam safely take at this section if the span is 18 ft. 0 ins. ?

$$b' = 8 \text{ ins.}$$
$$jd = \text{say } 14 \text{ ins.}$$
$$b'jd = 112 \text{ sq. ins.}$$

Limiting total shear $= 3 \times 75 = 225$ lbs. per sq. in.
on 112 sq. ins. or $225 \times 112 = 25{,}300$ lbs. total.

Shear strength of concrete

$$= 112 \times 75 = 8400 \text{ lbs.}$$

Shear strength of bent-up bars $= 2 \times 0{\cdot}785 \times \frac{1}{2} \times 18{,}000 = 14{,}100$ lbs.

Sectional area of each stirrup (two legs)

$$= 2 \times 0{\cdot}11 = 0{\cdot}22 \text{ sq. ins.}$$

Shear strength system

$$= \frac{14 \times 18{,}000 \times 0{\cdot}22}{4} = 13{,}900 \text{ lbs.}$$

Assuming that the concrete cracks under diagonal tension the combined strength of the bent-up bars and stirrups is $14{,}100 + 13{,}900 = 28{,}000$ lbs.

But this is more than the limiting total shear which fixes the maximum safe shear strength at

25,300 lbs.

Work out the strength in shear of the following.

(1) $d = 14''$, $j\,d = 12''$
¼" vertical stirrups at 6" centres as (a) Fig. 69.

(2) $d = 18''$, $j\,d = 15{\cdot}5''$
⅜" vertical stirrups at 8" centres as (b) Fig. 69.

(3) $d = 24''$, $j\,d = 21''$
⅜" vertical stirrups one pair every 5" centres as (c) Fig. 69.

(4) $d = 12''$, $j\,d = 10''$
¼" vertical stirrups as (b) Fig. 69 at 9" centres.

(5) $d = 10''$, $j\,d = 8''$
¼" vertical stirrups as (b) Fig. 69 at 10" centres.

(6) $d = 16''$, $j\,d = 14''$
⅜" vertical stirrups at 7" centres as (d) Fig. 69.

(7) A series of 1" diam. bars inclined at 30° and spaced horizontally at 12". $d = 25''$, $j\,d = 22''$. The bars are well anchored both ends.

(8) A single ¾" bar inclined at 25°, ends well anchored.

(1) Area of each stirrup in tension $= 2 \times {\cdot}049$ sq. ins.

$$\text{Shear} = \frac{12}{6} \times 2 \times {\cdot}049 \times 18{,}000 \text{ lbs.} = \textit{3530 lbs.}$$

(2) $$\text{Shear strength} = \frac{15{\cdot}5}{8} \times 2 \times {\cdot}11 \times 18{,}000 = \textit{7690 lbs.}$$

(3) $$\text{Shear strength} = \frac{21 \times 2 \times 2 \times {\cdot}11 \times 18{,}000}{5} = \textit{33,200 lbs.}$$

(4) The spacing $= {\cdot}75\,d$

$$\text{Strength say } \frac{1}{2} \times \frac{10 \times 2 \times {\cdot}049}{9} \times 18{,}000 = \textit{980 lbs.}$$

(5) The spacing $= d$

Strength in shear = nil.

(6) Shear strength $= \dfrac{14 \times 2 \times \cdot 11 \times 12{,}000}{7} = \mathit{5300\ lbs.}$

(7) This constitutes a system

$$\text{Shear strength} = \frac{\cdot 785 \times 18{,}000}{12} \times 22 \times (\cos 30^\circ + \sin 30^\circ) = \mathit{35{,}500\ lbs.}$$

(8) Shear strength $= \cdot 442 \times \sin 25^\circ \times 18{,}000 = \mathit{3360\ lbs.}$

Shear-Bond Stress.—It is clear from Fig. 62 that all shear problems involve bond stresses. Taking the bars in the short length δl there is a tension t on one end and a tension $t + \delta t$ on the other leaving an unbalanced pull δt. This must be taken up by bond stress between the bars and the concrete. The area of surface of the bars in this length is equal to their total perimeter multiplied by δl, and the bond stress is therefore :—

$$\text{Shear bond stress} = \frac{\delta t}{\delta l \times (\text{total perim.})}$$

$$= \frac{F}{jd \times (\text{total perim.})}$$

Thus, if at a given section $F = 14{,}000$ lbs. and $jd = 15$ ins. the tension reinforcement consisting of 2 bars 1 in. diam. the perimeter of each bar is 3·14 in. and

$$\text{shear-bond stress} = \frac{14{,}000}{15 \times (2 \times 3 \cdot 14)}$$

$$= 149 \text{ lbs. per sq. in.}$$

It is clear from standard practice that the calculated shear-bond stress may safely reach a figure of 200 lbs. per sq. in. for average 1 : 2 : 4 concrete (more for richer mixes).

It must clearly be understood that this high figure, which is twice the unit bond stress allowable in other cases, is only permissible as the *peak* value at the section of maximum shear. Whatever tensile stress there is in any bar at a given section there must always be a sufficient length of bar between the section and the free end of the bar to keep the *average* bond stress below about 100 lbs. per sq. in. Working to an allowable bond stress of 100 lbs. per sq. in. if the stress at any section is 4,000, 8,000, 12,000 or 16,000 lbs. per sq. in., we must have at least 10, 20, 30 or 40 bar diameters between that section and the end of the bar.

Example.—At a given section of a beam there is a total shear of 36,000 lbs. The lever arm $j\ d$ is 16 ins. and the tension steel consists of 6 bars $1\frac{1}{8}$ in. diam. What is the calculated shear-bond stress at this section?

$$\text{Shear-bond stress} = \frac{36{,}000}{16 \times 6 \times 3 \cdot 14 \times 1 \cdot 125}$$

$$= \mathit{106\ lbs.\ per\ sq.\ in.}$$

CHAPTER X

CONTINUOUS BEAMS

To calculate the moments and shears in a reinforced concrete beam is a difficult problem to attack directly. As a preliminary we shall take the simpler case of a perfectly uniform and elastic beam.

The results obtained from the analysis, although involving somewhat tedious working, are not open to doubt. We are able, therefore, to determine beyond all question the exact moments and shears at every point and the maximum and minimum values of these under any variable loading system that our ideal case can carry. We are next faced with the problem of comparing reinforced concrete beams as actually constructed with our ideal elastic beam. This problem varies from case to case but is in all cases largely a matter of conjecture. Extensive and costly experiments on full-sized structures might solve the problem for most cases that occur. We have, however, always one fact to rely on. In actual design certain approximate values of the main bending moments are taken and have been taken for many years and many thousands of existing beams have been constructed on such designs. These approximate values are therefore amply safe. Whether they could safely be reduced, future research and future construction alone will show.

Analysis of the Ideal Case.—The following conditions and properties are rigidly adhered to at all points.

(1) The beam has an infinite number of equal spans.

(2) The supports consist of frictionless, incompressible and inextensible points all at the same level.

(3) The value of EI is the same at all points and remains constant under all loads.

(4) The beam is very shallow compared with the length of the span and can be conveniently represented by its centre line.

(5) The inclination of this centre line to the horizontal is small so that we can write $\frac{d^2 y}{d x^2} = \frac{1}{\rho}$ without error (where ρ is the radius of curvature).

(6) The beam itself has no weight and when unloaded lies in a perfectly straight line without initial stress.

(7) The supports at the extreme ends of the beam are free, i.e., there is never any bending moment at these points.

(8) There are no end thrusts.

Nature of the Problem.—We shall first of all consider the case where the beam supports a single load at one point. Having found the moments and shears produced we can draw influence lines (by varying the position of the point). Integration of the influence line areas will give bending moments under distributed load.

To find the maximum positive bending moment at a certain point, under any distribution of load, it is necessary to draw the bending moment influence line for that point and then load all those parts of the beam for which the influence line is positive as heavily as possible. The moment produced by this loading can be obtained by summation or integration. By choosing all the negative areas we can obtain the minimum bending moment (i.e., the maximum negative moment).

Those moments which produce convexity on the under side of the beam are called positive as for such moments $\frac{d^2 y}{d x^2}$ is positive when x is measured from left to right.

Lemma I.—To find the deflection of a cantilever under a point load.

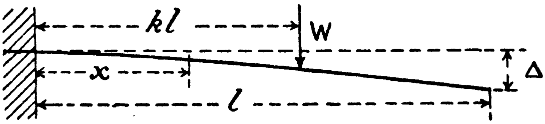


Fig. 70.

Fig. 70 represents a cantilever carrying a load W distant $k\,l$ from the support.

Since there is no load between W and the free end, there is no moment on this portion and this part of the beam is straight.

The total deflection $\triangle$

$$= -\frac{W\,(k^3\,l^3)}{3\,EI} - l\,(1-k)\cdot\frac{W\,(k\,l)^2}{2\,EI}$$

extending the case to a freely supported beam.

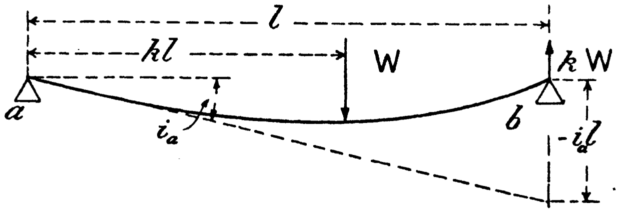


Fig. 71.

Suppose the slope at $a = i_a$. Then the beam may be treated as a cantilever carrying a point load W and an end load kW. The total " deflection " reckoned from the tangent to the slope at $a = -\,l i_a$.

Deflection upwards due to kW (a well-known result)

$$= \frac{Wk\,l^3}{3\,EI}.$$

Deflection downwards due to W

$$= \frac{W\,(k\,l)^3}{3\,EI} + l\,(1-k)\,\frac{W\,(k\,l)^2}{2\,EI} \text{ as shown above.}$$

The difference $= -\,l\,i_a$ (negative if i_a is positive)

$$= \frac{Wk\,l^3}{3\,EI} - \frac{W\,(k\,l)^3}{3\,EI} - l\,(1-k)\,.\,\frac{W\,(k\,l)^2}{2\,EI}$$

$$= \frac{W\,k\,l^3}{6\,EI}\,[k^2 - 3\,k + 2] \text{ for a freely supported beam.}$$

Lemma II.—To find the deflection of a beam under "fixing moments" only.

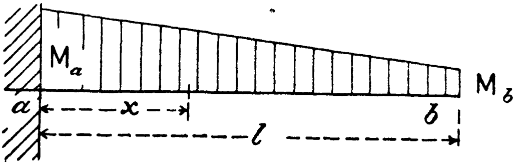


Fig. 72.

Fig. 72 represents a beam of span l the bending moment at a being M_a and at b being M_b. Since there is no load on the beam the shear force is constant and the bending moment varies uniformly.

Assuming that $\frac{d\,y}{d\,x}$ and y are zero at a

when $x = l$ we have

$$\frac{dy}{dx} = \frac{1}{E\,I}\left(M_a\,\frac{l}{2} + M_b\,\frac{l}{2}\right)$$

$$y = \frac{1}{E\,I}\left(M_a\,\frac{l^2}{3} + M_b\,\frac{l^2}{6}\right)$$

for a beam under " fixing moments " only.

Main Problem. Centre spans.—Let Fig. 73 represent our ideal beam having an infinite number of spans all of which $= l$. The only load is W on span ab.

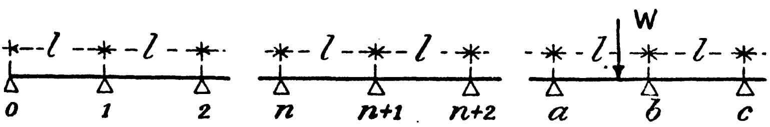


Fig. 73

The points of support are numbered 0, 1, 2, 3, 4, etc., n, $n + 1$, $n + 2$, etc.

a, b and c are three points of support in the centre of the whole beam such that there are an infinite number of spans to the left of point a and also to the right of point c.

Take any three consecutive points of support n, $n + 1$, and $n + 2$.

By the theorem of three moments we have, since there is no load on either span,

$$M_n + 4\,M_{n+1} + M_{n+2} = 0.$$

Also we have $\quad M_0 = 0$ (free end)

$\therefore \quad M_2 = -\,4\,M_1$

and $\quad M_3 = +\,15\,M_1$

and by applying our equation we can eventually express the bending moments at all points of support as far as point a in terms of M_1 by the series

$$M_n = (-\,1)^{n+1}\,\frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}} \times M_1\,{}^{*}.$$

This last expression is the most important in this analysis, and having solved it the results soon follow.

Limit when n tends to infinity ,

$$\text{of} \quad \frac{M_{n+1}}{M_n} = \text{Limit of} - \frac{(2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}}{(2+\sqrt{3})^n - (2-\sqrt{3})^n}$$

as n tends to infinity $(2 - \sqrt{3})n$ tends to zero thus,

$$\text{Limit of } \frac{M_{n+1}}{M_n} = -\;(2 + \sqrt{3})\ {}^{**}.$$

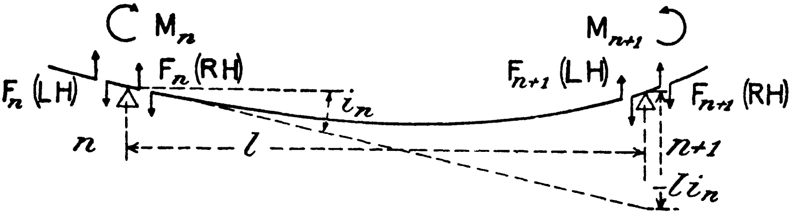


Fig. 74.

Consider the span between support (n) — $(n + 1)$ (unloaded).

Let the bending moment at n be M_n ; (Fig. 74)

Let the slope at n be i_n and the shears be F_n (L.H.) immediately to the left of n and F_n (R.H.) immediately to the right of n.

Similarly let M_{n+1} be the moment at $(n + 1)$, etc. Total deflection on span $= -\,li_n$.

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[*] Published in *Engineering*, Aug. 1914.

[**] It must be clearly understood that this applies to the case where there are an infinite number of unloaded spans to the left of n and to this particular case only.

The reason for the negative sign is that if i_n is positive, the upward (positive) deflection $= -li_n$.

In the figure i_n has been drawn negative.

By Lemma II

$$\text{Total deflection} = \frac{1}{EI}\left(M_n \times \frac{l^2}{3} + M_{n+1} \times \frac{l^2}{6}\right)$$

and

$$\frac{M_{n+1}}{M_n} = -(2 + \sqrt{3})$$

$$\therefore \quad -li_n = \frac{1}{EI}\left(M_n \times \frac{l^2}{3} - (2 + \sqrt{3})\, M_n \times \frac{l^2}{6}\right)$$

$$\therefore \quad i_n = +\frac{M_n \times l}{2\sqrt{3}\,EI} \text{ (when } n \text{ tends to infinity)}$$

This may be looked upon as the " end condition " in our ideal case and should be compared with

$$i_n = 0 \text{ for an encastré beam.}$$

Also by taking moments we have :—

$$F_n\,(\text{R.H.}) \times l + M_{n+1} = M_n$$

giving

$$F_n\,(\text{R.H.}) = +\frac{(3 + \sqrt{3})\, M_n}{l}.$$

Similarly considering the span $(n - 1)$ to (n)

$$F_n\,(\text{L.H.}) = +\frac{(3 + \sqrt{3})}{l}\, M_{n-1} = -\frac{(3 - \sqrt{3})\, M_n}{l}.$$

$$\begin{aligned} \text{Reaction at } n &= F_n\,(\text{L.H.}) - F_n\,(\text{R.H.}) \\ &= -\frac{(3 - \sqrt{3})\, M_n}{l} - \frac{(3 + \sqrt{3})\, M_n}{l} \\ &= -\frac{6\, M_n}{l} \text{ when } n \text{ tends to infinity.} \end{aligned}$$

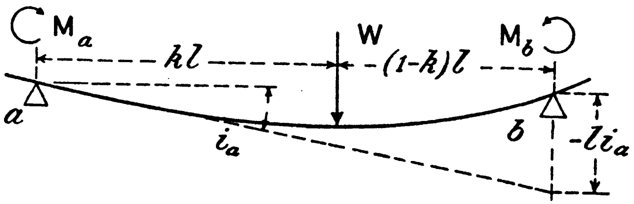

Fig. 75.

Loaded Span.—Suppose our single load W is placed $k\,l$ from a.

We can imagine the bending moment on the span as consisting of two portions. (Fig. 75.)

(1) A bending moment caused by W on a freely supported span l.

(2) Fixing moments due to M_a and M_b as in Lemma II.

Upward deflection of span $= - i_a l$ (as before).

$= $ (upward deflection due to bending moment (1))

$+$ (upward deflection due to bending moment (2))

$$= \frac{W k l^3}{6\,EI}\,[k^2 - 3k + 2] + \frac{1}{EI}\left[M_a \frac{l^2}{3} + M_b \frac{l^2}{6}\right].$$

The first part from Lemma I and the second part from Lemma II.
The total slope change from a to b

$$= \text{area of B.M. diagram} \times \frac{1}{E\,I}$$

$$= \frac{Wk\,(1-k)\,l^2}{2\,EI} + \frac{1}{E\,I}\left(M_a \times \frac{l}{2} + M_b \times \frac{l}{2}\right)$$

$$= i_b - i_a.$$

Since there are an infinite number of unloaded spans to the left of a and also to the right of b

We have $i_a = + \dfrac{M_a \times l}{2\sqrt{3}\,EI}$

and $i_b = - \dfrac{M_b \times l}{2\sqrt{3}\,EI}$ (allowing for the change in direction)

$$i_b - i_a = -\frac{M_b \times l}{2\sqrt{3}EI} - \frac{M_a \times l}{2\sqrt{3}EI} = \frac{W\,k(1-k)l^2}{2\,EI} + \frac{1}{EI}\left(M_a \times \frac{l}{2} + M_b \times \frac{l}{2}\right)$$

i.e., $$-\frac{(M_a + M_b)}{2\sqrt{3}} = \frac{W\,k\,(1-k)l}{2} + \frac{M_a + M_b}{2}$$

$$M_a + M_b = -\frac{W\,k\,(1-k)\,l\,(3-\sqrt{3})}{2}$$

$$= \frac{W\,k\,(k-1)\,l\,(3-\sqrt{3})}{2} \quad \ldots\ldots \quad (1)$$

From the equation for the deflection by writing i_a in terms of M_a.
We have

$$-\frac{M_a \times l^2}{2\sqrt{3}\,EI} = \frac{W\,k\,l^3}{6\,EI}\,[k^2 - 3\,k + 2] + \frac{1}{EI}\left[M_a \times \frac{l^2}{3} + M_b \times \frac{l^2}{6}\right]$$

giving $$-\frac{M_a}{2\sqrt{3}} = \frac{W\,k\,l\,(k-1)\,(k-2)}{6} + \frac{M_a}{3} + \frac{M_b}{6}$$

$$-\sqrt{3}\,M_a - 2\,M_a - M_b = W\,k\,l\,(k-1)\,(k-2)$$

$$M_a\,(2 + \sqrt{3}) + M_b = -\,W\,k\,l\,(k-1)\,(k-2)$$

also $$M_a \qquad + M_b = \quad W\,k\,l\,(k-1)\,\frac{(3-\sqrt{3})}{2} \quad \ldots (1)$$

$$M_a\,(1 + \sqrt{3}) = -\,Wk\,l\,(k-1)\left(k - 2 + \frac{(3-\sqrt{3})}{2}\right)$$

$$M_a = -\frac{W\,k\,l}{2}(1-k)\,(1+(1-\sqrt{3})\,k)$$

writing $(1 - k)$ for k we have

$$Mb = -\frac{W\,k\,l}{2}(1-k)\,(1+(1+\sqrt{3})\,k)\,(2-\sqrt{3}).$$

Thus for any position of the load W on the span $a\ b$ we can calculate M_a and M_b. Thus we know at once the bending moments at all points of that span. We have already shown that the bending moments at successive points of support die away in the ratio $-(2+\sqrt{3})$ as we get farther away from the loaded span. We can thus draw the bending moment diagram for the whole number of spans.

For example let $k = \cdot 5$ (i.e., the load W is at mid-span)

$$M_a = -\frac{W \times \cdot 5 \times l}{2} \times \cdot 5 \times \left(1 + \frac{(1-\sqrt{3})}{2}\right)$$

$$= -\frac{W\,l}{16} \times (3-\sqrt{3}) = -\cdot 0791\ W\,l$$

$$M_b = -\frac{W \times \cdot 5 \times l}{2} \times \cdot 5 \times \left(1 + \frac{1+\sqrt{3}}{2}\right) \times (2-\sqrt{3})$$

$$= -\frac{W\,l}{16} \times \frac{(3+\sqrt{3})}{(2+\sqrt{3})} = -\cdot 0791\ W\,l.$$

The bending moment under the load itself

$$= \frac{Wl}{4} - \cdot 0791\ W\,l = (\cdot 25 - \cdot 0791) = \cdot 1709\ W\,l$$

The bending moment at $c = -\dfrac{1}{(2+\sqrt{3})} \times -\cdot 0791\ W\,l$

$$= +\cdot 0213\ W\,l.$$

Fig. 76 shows the bending moment diagram for this case.

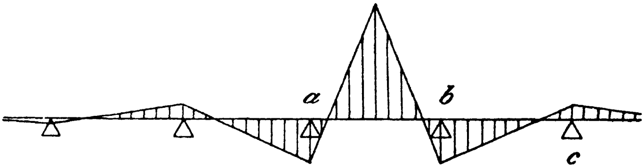

Fig. 76

Suppose we write $k = \cdot 1, = \cdot 2, = \cdot 3, = \cdot 4, = \cdot 5$, etc., we can tabulate M_a and M_b for any number of values.

$k = \cdot 1,\ M_a = -\cdot 0419\ W\,l,\ M_b = -\cdot 0153\ W\,l$
$\cdot 2,\ M_a = -\cdot 0682\ W\,l,\ M_b = -\cdot 0331\ W\,l$
$\cdot 3,\ M_a = -\cdot 082\ W\,l,\ M_b = -\cdot 0511\ W\,l$

$k = \cdot 4,\ M_a = -\ \cdot 085\,W\,l,\ M_b = -\ \cdot 0677\,W\,l$
$\cdot 5,\ M_a = -\ \cdot 0791\,W\,l,\ M_b = -\ \cdot 0791\,W\,l$
$\cdot 6,\ M_a = -\ \cdot 0677\,W\,l,\ M_b = -\ \cdot 085\,W\,l$

and so on symmetrically.

Let us take a case and work out the moments on the span *a b* put $k = \cdot 6$ then $M_a = -\ \cdot 0677\,W\,l$ and $M_b = -\ \cdot 085\,W\,l$.

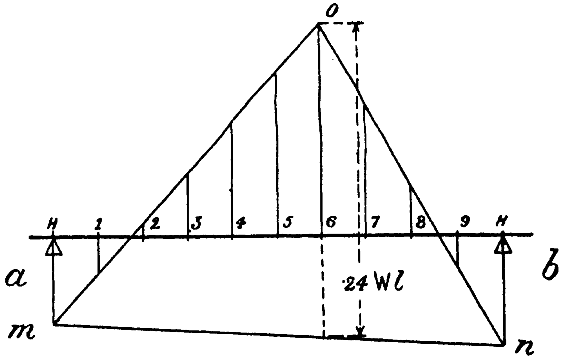

Fig. 77.

Set off these values in Fig. 77 and join the line *m n*.

Set up the triangle *m o n* to represent the " free " bending moment, i.e., the bending moment that would result if the beam were freely supported at *a* and *b*.

This $= W\,k\,(1 - k)\,l = W \times \cdot 6 \times \cdot 4\,l = \cdot 24\,W\,l$.

At any point then we have

(I) a positive " free " moment

(II) a negative " fixing " moment.

A little arithmetic will give us the values at the 10 points shown.

Thus

For Point	1	2	3	4	5	6	7	8	9
+ *ve M*	·04	·08	·12	·16	·20	·24	·18	·12	·06 W
− *ve M*	·0694	·0712	·0729	·0746	·0763	·0781	·0798	·0815	·0833 W
Net. M	−·0294	+·0088	+·0471	+·0854	+·1237	+·1619	+·1002	+·0385	−·0233 W

By repeating the process for other values of *k* we obtain Table CB I.

We have now only to deal with the moments caused by loads on other spans and we can draw out our influence lines for the ten points which divide the span into ten equal parts.

Suppose we have a load *W* on span *b c* distant ·6 *l* from *b* (Fig. 78).

Since the span *b c* is exactly similar in all its properties to the span *a b*.

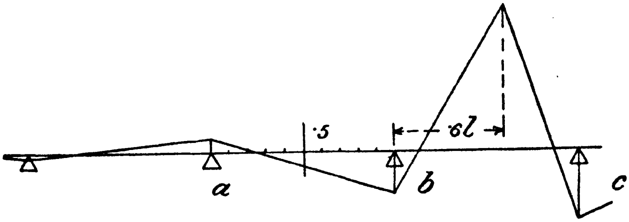

Fig. 78.

We have $M_b = -\cdot0677\,W\,l$ (from Table CB I).

$$M_a = -\frac{Mb}{2+\sqrt{3}} = +\cdot0181\,W\,l.$$

Since the ratio $\frac{M_a}{M_b}$ is a constant for any load on the span $b\ c$ the bending moments at the ten points on span $a\ b$ will all be some fixed fraction of M_b.

Thus $M_a = -\frac{M_b}{(2+\sqrt{3})} = -M_b\,(2-\sqrt{3}) = -\cdot268\,M_b$.

Bending moment at point 5 $= \frac{1}{2}(M_a - M_b)$

$= \frac{1}{2}\,M_b\,(1 - \cdot268) = \cdot366\,M_b$.

$\therefore$ The bending moment at point 5 fot a load as shown in Fig. 78

$$= \cdot366\,M_b = \cdot366 \times -\cdot0677\,W\,l$$
$$= -\cdot0248\,W\,l$$

Furthermore if the load W were on the span $c\ d$ distant $\cdot6\,l$ from c.

Then M_c would be $-\cdot0677\,W\,l$

M_b would be $-(2-\sqrt{3}) \times (-\cdot0677)\,W\,l$

and the bending moment at point 5 would be still

$$\cdot366 \times M_b = \cdot366 \times (-(2-\sqrt{3})) \times (-\cdot0677)\,W\,l$$
$$= +\cdot00664\,W\,l.$$

Influence Lines for Bending Moments.—If we can draw the bending moment diagram for any position of the load W it follows that we can draw the bending moment influence line for any point. A bending moment diagram shows the bending moment at every point for one particular position of the load. An influence line shows the bending moment produced at one particular point by every possible position of the load. As an example we will construct the influence line for a point of support.

When the load W is directly over a point of support the load passes directly on to that support and no shears or bending moments are produced. Therefore all bending moment influence lines have a zero value at all points of support.

We will take the support a

when the load is directly over a the moment at $a = 0$.

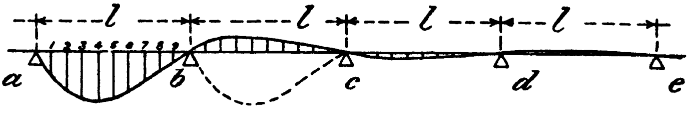

Fig. 79.

From Table CB I when the load is on point 1 the bending moment at $a = -\cdot 0419\ W\ l$.

Therefore at point 1 Fig. 79 set down a distance equal to $-\cdot 0419\ W\ l$.

When the load W is on point 2 the bending moment at $a = -\cdot 0682\ Wl$. Therefore at point 2 set down a distance equal to $-\cdot 0682\ W\ l$.

We see at once that the horizontal lines of figures on Table CB I give us when plotted the bending moment diagrams while the vertical columns of figures give us the influence lines.

We thus have all the values necessary between a and b.

When the load passes b and rolls on to the span $b\ c$ we have the following fact.

$$M_a = -\frac{M_b}{(2+\sqrt{3})}.$$

If now we draw the influence line for the point b shown dotted in Fig. 79 and multiply all its ordinates by $-\frac{1}{(2+\sqrt{3})}$ we shall have the influence line for point a as shown.

For example when the load W is on point 5 the bending moment at $a = -\cdot 0791\ Wl$ (Table CB I). Therefore when the load is mid-way between b and c the bending moment at $b = -\cdot 0791\ W\ l$

$$\text{and } M_a = -\frac{M}{2+\sqrt{3}} = -\frac{1}{(2+\sqrt{3})} \times -\cdot 0791\ W\ l = +\cdot 0214\ Wl.$$

Thus the influence line for M_a for positions of the load W between b and c is exactly similar to the influence line for positions of the load between a and b but all the ordinates are reduced in the ratio of $1 : \frac{-1}{2+\sqrt{3}}$. Similarly for positions of the load between c and d the influence line is again exactly similar but all the ordinates are reduced in the ratio $1 : \frac{+1}{(2+\sqrt{3})^2}$.

Fig. 79 shows the influence lines for positions of W to the right of a. By symmetry the influence line for positions to the left of a follows at once.

Plate I shows this influence line to a large scale for four consecutive centre spans.

The method of constructing the influence line for any other point is similar to the method described for the point *a*.

Let us take the point marked 5 on Fig. 80.

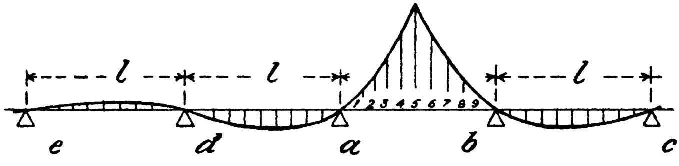

Fig. 80.

Referring to Table CB I, we have all the values for the span *a b* in the vertical column headed mid-span.

The maximum value of the bending moment at point 5 occurs when W is directly over the point and is equal to $+ \cdot 1709\ W\ l$. Suppose now that the load W passes b and is on the span $b\ c$. By Fig. 78 we see the bending moment at point 5 is always a definite fraction of the bending moment at b.

By similar triangles this value

$$= M_b\,(1 - \cdot 5\,(3 - \sqrt{3})\,) = M_b\,(1 - \cdot 634) = \cdot 366\ M_b.$$

For positions of W on the span *d a* we have similarly, bending moment at point 5

$$= M_a\,(1 - \cdot 5\,(3 - \sqrt{3})\,) = M_a\,(1 - \cdot 634) = \cdot 366\ M_a.$$

Referring to Plate I we find this influence line marked *E*. On this plate the influence lines for the points marked *H* (haunch) *A*, *B*, *C*, *D* and *E* (mid-span) have been drawn. It is obvious at a glance which point each line refers to as there is a distinct peak where the influence line crosses the point to which it refers, all other part of the lines being smooth curves.

All points for the lines in the right-hand centre span are given in Table CB I. All other points are obtained by taking the line marked *H* in the right-hand centre span and multiplying all its ordinates by the appropriate numerical factors.

Thus by multiplying these ordinates by ·366 in accordance with the analysis immediately above we obtain the values for line *E* in the extreme right-hand span on Plate I. By multiplying by ·366 and drawing them opposite-handed we obtain the values for line *E* in the left-hand centre span on Plate I.

Application of Plate I.—For office use Plate I must be drawn out by the reader himself on tracing cloth. The author uses a copy which is 40 ins. long, i.e., $L = 10$ ins. and 7 ins. high and finds this a convenient size. All work should be done on a blue-print copy of the plate as the designer can then mark any points he requires or scribble on any notes. When in course of time the print becomes torn or too much scribbled over a fresh print is easily obtained.

Using the values given in Table CB I the construction of such a tracing is not a long job. If there is a series of rolling loads these loads should be marked on a thin strip of cartridge paper to the same linear scale as the blue print. For example should the actual span in question be 25 ft. 0 ins. while the length of L on the blue-print of Plate I scales 9·8 ins. (all prints are apt to shrink) then 9·8 ins. on the plate represents 25 ft. 0 ins. linear. If two wheel loads 6 ft. 0 ins. apart occur on the span then two marks $\frac{6}{25} \times 9\cdot8$ ins. apart on the strip of cartridge paper would represent them. By sliding the strip of cartridge paper which represents a scale model of the rolling load over the span we can obtain by trial and error the maximum bending moment at any point.

Take an easy example. A gantry crane is supported on a continuous girder which has a large number of equal spans all 26 ft. 0 ins. long. The load from the crane trolley may be taken as a single point load of 15 tons. If $E\ I$ is constant what are the following ?

(a) the maximum positive moment at mid-span
(b) „ „ negative „ „ „
(c) „ „ „ „ „ the supports
(d) „ „ positive „ „ „ „

for the middle spans.

From Plate I

(a) occurs when W is at mid span $= + \cdot1709\ WL$
(b) occurs when W is on ·4 in the next span and $= - \cdot03\ WL$
(c) occurs when W is ·4 L from the support and $= - \cdot085\ WL$
(d) occurs when W is ·4 L from the next point of support and $= + \cdot023\ WL$

since $W = 15$ tons, $L = 26$ feet
(a) $= + \cdot1709 \times 15 \times 26 = 67$ tons-ft.
(b) $= - \cdot03 \times 15 \times 26 = - 11\cdot7$ „
(c) $= - \cdot085 \times 15 \times 26 = - 33$ „
(d) $= + \cdot023 \times 15 \times 26 = + 9$ „

Notice the small numerical value of the moment caused at the support. If the span had been 26 ft. 0 ins. encastré the bending moment at the support would have had a maximum negative value of about $- \cdot150\ W\ l$, i.e., nearly twice the value above.

End Spans Lemma III.—To find the effect of a bending moment applied to the free end of an unloaded system :

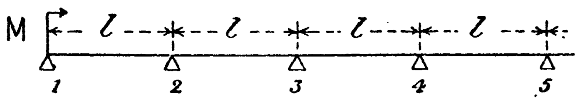

Fig. 81.

Suppose 1, 2, 3, 4, are the points of support, point 1 being the free end.

Apply a moment M to the point 1.

We have therefore $M_1 = M$.

Also since there are an infinite number of unloaded spans to the right of point 1

$$M_2 = -\frac{1}{(2+\sqrt{3})}M_1$$

$$M_3 = +\frac{1}{(2+\sqrt{3})^2}M_1 \text{ etc.}$$

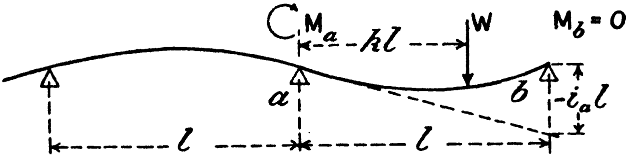

Fig. 82.

End Span.—Let $a\ b$ be the end span of an infinite number of equal spans then $M_b = 0$ as b is a free end.

Let W be the only load on the whole system distant $k\ l$ from a.

Since there are an infinite number of unloaded spans to the left of a we have

$$i_a l = \frac{M_a l^2}{2\sqrt{3}\,EI} \text{ as before.}$$

The bending moment on the span $a\ b$ may be divided into two parts.

(I) The moment caused by W on a freely supported beam

(II) The moment due to the " fixing moment " M_a.

If the slope at $a = i_a$.

Total deflection upward $= -\ i_a\ l$.

Deflection upward due to (I)

$$= \frac{W\,k\,l^3}{6\,EI}\,[k^2 - 3\,k + 2] \text{ by Lemma I.}$$

Deflection upward due to (II)

$$= \frac{M_a l^2}{3\,E\,I} \text{ by Lemma II.}$$

$$\therefore \qquad -\ i_a\,l = \frac{W\,k\,l^3}{6\,EI}\,[k^2 - 3\,k + 2] + \frac{M_a\,l^2}{3\,EI}$$

$$= \text{also } -\frac{M_a\,l^2}{2\sqrt{3}\,EI}\,.$$

$$\therefore \qquad -\frac{M_a l^2}{2\sqrt{3}\,EI} = \frac{W k l^3}{6\,EI}[k^2 - 3k + 2] + \frac{M_a l^2}{3\,EI}$$

$$\therefore \quad M_a = -\frac{Wkl(1-k)(2-k)}{(2+\sqrt{3})}.$$

By putting $k = \cdot 1, = \cdot 2, = \cdot 3$, etc., we have values of M_a for all values of k.

Thus if $k = \cdot 4$.

$$M a = -\frac{W \times \cdot 4\, l \times \cdot 6 \times 1\cdot 6}{(2 + 1\cdot 732)} = -\cdot 104\, W l.$$

The value of M_a can also be determined by the use of Lemma III in conjunction with the analysis for the centre spans.

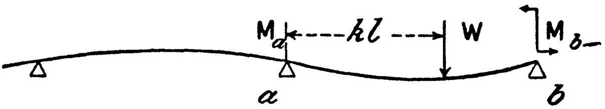

Fig. 83.

In Fig. 83 let $a\ b$ be the end span under a load W.

Apply a moment at b equal to M_b.

Now the moment at a is caused by

(I) The bending moment due to W only with b " free."

(II) The bending moment due to the applied moment M_b.

We have

(Modified M_a) = (M_a with b " free ") + (change due to M_b).

$\therefore$ by Lemma III

$$(\text{Modified } M_a) = (M_a \text{ with } b \text{ `` free ''}) + \left(\frac{-M_b}{(2+\sqrt{3})}\right).$$

If we make $M_b = -\frac{Wkl}{2}(1-k)(1+(1+\sqrt{3})k)(2-\sqrt{3})$

which would be the value of M_b if we added an infinite number of spans to the right of b thus turning ab into a centre span, then M_a must equal the value of M_a for a centre span

thus $\quad (\text{Modified } M_a) = -\frac{Wkl}{2}(1-k)(1+(1-\sqrt{3})k)$

$$\therefore -\frac{Wkl}{2}(1-k)(1+(1-\sqrt{3})k) = (M_a \text{ with } b \text{ `` free ''})$$
$$+\frac{Wkl}{2}\,\frac{(1-k)(1+(1+\sqrt{3})K)(2-\sqrt{3})}{(2+\sqrt{3})}.$$

$\therefore$ M_a with end b " free "

$$= -\frac{Wkl(1-k)}{2}\left(1 + (1-\sqrt{3})k + (7-4\sqrt{3})(1+(1+\sqrt{3})k)\right)$$

$$\therefore M_a = -\frac{Wkl(1-k)(2-k)}{(2+\sqrt{3})} \text{ with } b \text{ `` free ''}$$

as already obtained.

This method of applying an imaginary bending moment at b is a great time-saver when calculating the influence lines and envelopes for those spans next to the end span.

If we know M_a we can readily obtain the values of the bending moments at other parts of the end span.

Table CB II shows in a similar manner to Table CB I the bending moment diagrams and influence lines for the end span itself.

These values have been plotted on Plate II.

Coming next to the span next to the end.

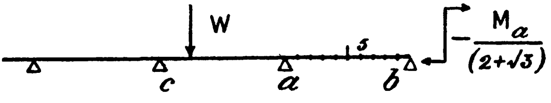

Fig. 84.

Apply an imaginary moment $-\dfrac{M_a}{(2+\sqrt{3})}$ at point b.

Then the conditions revert to centre spans.

The change in M_a due to adding the moment at b

$$= -\frac{M_b}{(2+\sqrt{3})} = +\frac{M_a}{(2+\sqrt{3})^2}.$$

Thus we have

$$(M_a \text{ for centre spans}) = (M_a \text{ for end span}) + \frac{(M_a \text{ centre span})}{(2+\sqrt{3})^2}.$$

$$\therefore \quad M_a \text{ for end span} = (M_a \text{ for centre spans})\left(1 - \frac{1}{(2+\sqrt{3})^2}\right)$$
$$= \cdot 928 \; (M_a \text{ for centre spans}).$$

The bending moments caused at points between a and b by a load W on span ac are easily found, for since b is free they are all simple fractions of M_a.

Referring to Plate II the influence lines shown for the support next to the end support is line L for the point of support a in Fig. 84. As already shown this is exactly similar to the line marked H for the two centre spans on Plate I except that all the ordinates are reduced in the ratio 1 : ·928.

The lines marked B, D, F and H in the span next to the end span on Plate I, are respectively drawn to have $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$ the ordinates of the line marked L.

Following a somewhat similar procedure it can be shown that the line marked L in the left hand span on Plate II is similar to the line marked H in the left-hand span on Plate I except that its ordinates are increased in the ratio 1 : 1·02.

Application of Plate II.—The use of Plate II is exactly similar to that of Plate I. It must be drawn out on tracing cloth to some convenient scale (the author uses one having $L = 10$ ins. which is 7 ins. high overall).

These two influence lines are quite general in their application and will give by trial and error the maximum bending moment at any point under any system of point loads. There are, however, one or two particular cases which are worthy of a separate mention.

Particular Cases. Envelopes of Moment.—Suppose we have a main beam of many equal spans carrying a series of secondary beams one of which occurs at the middle of each span of the main beam. Neglecting the dead weight of the structure we may assume that any one of these secondary beams is either loaded with superload or unloaded. The problem is to find what arrangement of loaded and unloaded beams will create the maximum possible bending moment at all points. Take first of all the centre spans.

Referring to Plate I we have at once the arrangement which will give us the maximum moment for any point because we can see by following the influence line for any point exactly which loads will cause positive and which will cause negative moments. By choosing all the mid-points which give positive moments and loading them fully we obtain the arrangement of loads giving the maximum positive moment.

Take for example the support *d* in Fig. 92 (1). We see from Plate I that a load *W* placed in the centres of spans *ab*, *cd*, *de* and *fg* will produce negative moment at the point of support *d*. Also in Fig. 92 (2) point loads *W* at mid-span in spans *bc* and *ef* produce positive moment at the support *d*.

Therefore by loading *cd* and *de* missing *bc* and *ef*, loading *ab* and *fg*, missing the next spans, etc., ad infinitum, we have the arrangement giving the greatest negative moment, whereas by loading as drawn in Fig. 92 (2) we have the greatest positive moment at *d*.

From Table CB I a point load in the centre of span *cd* or *de* produces a moment.

$$M_d = -\cdot 0791\,W l = -\frac{W l}{16}(3-\sqrt{3}).$$

Also a point load in the centre of spans *bc* or *ef* produces a moment

$$M_d = +\frac{W l}{16}\,\frac{(3-\sqrt{3})}{(2+\sqrt{3})} = \cdot 0213\,W l.$$

A point load in the centre of span *ab* or *fg* produces a moment

$$M_d = -\frac{W l}{16}\,\frac{(3-\sqrt{3})}{(2+\sqrt{3})^2} = -\cdot 0057\,W l.$$

Greatest negative moment produced by arrangement 1 Fig. 92

$$M_d = -\frac{W l}{16}(3-\sqrt{3}) \times 2\left[1+\frac{1}{(2+\sqrt{3})^2}+\frac{1}{(2+\sqrt{3})^4}+\dots\right].$$

Summing the G. P. to infinity

$$M_d = -\frac{W l}{16}(3-\sqrt{3})\times 2\,\frac{(3+2\sqrt{3})}{6} \doteq -\frac{W l}{16}(\sqrt{3}+1) = -\cdot 1709\,W l.$$

Greatest positive moment

$$M_d = + \frac{W\,l}{16}(3 - \sqrt{3}) \times 2\left[\frac{1}{(2 + \sqrt{3}} + \frac{1}{(2 + \sqrt{3})^3} + \ldots .\right]$$

$$= \frac{W\,l}{16}(\sqrt{3} - 1) = + \cdot 0458\,W\,l.$$

Suppose now we have all the centre points loaded, by comparing with the case of an encastré beam we see that

$$M_d = - \cdot 125\,W\,l.$$

We should be able to obtain this result by adding case (1) to case (2)

$$-\frac{W\,l}{16}(\sqrt{3} + 1) + \frac{W\,l}{16}(\sqrt{3} - 1) = \frac{W\,l}{8}$$

which is obviously correct.

Coming next to the centre point. From Plate I it appears that arrangement (4) in Fig. 92 gives the greatest positive moment while arrangement (5) gives the greatest negative moment at point *h*.

From Table CB I

$$M_d \text{ for a load at } h \text{ only} = - \frac{W\,l}{16}(3 - \sqrt{3}) = - \cdot 0791\,W\,l.$$

M_d for arrangement (4) in Fig. 92.

$$= - \frac{W\,l}{16}(3 - \sqrt{3})\left[1 - \frac{1}{(2 + \sqrt{3})} + \frac{1}{(2 + \sqrt{3})^2} - \ldots .\right]$$

$$= - \frac{Wl}{16}(3 - \sqrt{3}) \times \frac{(2 + \sqrt{3})}{(3 + \sqrt{3})} = - \frac{W\,l}{16}.$$

Also $\qquad M_e = - \dfrac{Wl}{16}$

$$M_h = + \frac{W\,l}{4} - \frac{W\,l}{16} = + \cdot 1875\,W\,l.$$

M_d for arrangement 5 in Fig. 92.

$$= - \frac{W\,l}{16}(3 - \sqrt{3})\left[1 - \frac{1}{(2 + \sqrt{3})} + \frac{1}{(2 + \sqrt{3})^2} - \ldots .\right]$$

$$= - \frac{W\,l}{16}.$$

Also $\qquad M_e = M_d = M_h = - \cdot 0625\,W\,l.$

From Plate I it appears that the influence lines for points *C*, *D* and *E* are always positive when line *E* is positive. Arrangements 4 and 5 therefore give the maximum possible moments at points *C* and *D* as well as at point E.

The influence line for *A*, considering a mid-span load only, has the same sign as line *H*. Therefore the arrangements (1) and (2) apply to point *A*. The line for point *B* however is the same sign as *E* for some parts and the same sign as H for others and thus requires a further arrangement (not given here).

Table CB III gives these values.

Coming now to the end span we see from Plate II that arrangement (6) in Fig. 92 gives the greatest possible negative moment.

From Lemma III we see that if we apply a Moment M_s to the free end s

Such that $M_s = M_e$ in case (1) on same figure then M_r would equal M_d in case (1).

Therefore M_r (with end free)

$$= M_d \text{ in case (1)} + \frac{1}{(2+\sqrt{3})} M_e$$

$$= -\frac{Wl}{16}(\sqrt{3}+1) - \frac{1}{(2+\sqrt{3})} \times \frac{Wl}{8}(2-\sqrt{3})$$

$$= -\frac{Wl}{16}(15 - 7\sqrt{3}) = -\cdot 180\, W\, l.$$

Similarly for case (7).

$$M_r = -W\, l \times \cdot 125 \left(1 + \frac{1}{(2+\sqrt{3})}\right)$$

$$= -W\, l \times \cdot 125\,(3 - \sqrt{3}) = -\cdot 1585\, W\, l.$$

The further analysis for the end span will not be given but the values are shown in Table CB III.

Tables CB IV gives similar values for the case where two secondaries occur in each span. The analysis for this case will not be given, also the exact values of the bending moments between ·1 and ·25 have not been worked out.

Envelope for System under a Superload.—The term superload is meant to indicate a uniformly distributed load which may occur on any part of the beam while any other part is unloaded. We shall proceed to find the maximum possible value of the bending moments produced at all parts of the system.

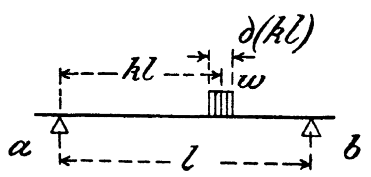

Fig. 85.

Let ab Fig. 85 be a centre span.

Imagine that a short length of beam $\delta\,(kl)$ is loaded with a uniform load of w per unit length.

Total load on beam $= w \times \delta\,(kl)$.

If $\delta\,(kl)$ is small we can treat this as a point load, we have therefore as before

$$M_a = -w\,\delta\,(kl) \times \frac{l}{2} \times k\,(1-k)\,(1 + (1-\sqrt{3})\,k)$$

$$= - w\,\delta\,(kl) \times \frac{l}{2}[k - \sqrt{3}\,k^2 + (\sqrt{3} - 1)\,k^3].$$

Since l is constant $k\,\delta\,l = \delta\,(kl)$

we have $\quad M_a = -\frac{wl^2}{2}\,[k - \sqrt{3}\,k^2 + (\sqrt{3} - 1)\,k^3]\,\delta\,k.$

If the whole span ab were loaded

$$\text{Total} \quad M_a = -\int_{k=0}^{k=1} \frac{wl^2}{2} \quad [k - \sqrt{3}\,k^2 + (\sqrt{3} - 1)\,k^3]\,\delta\,k$$

$$= -\frac{wl^2}{2}\left[\frac{k^2}{2} - \frac{\sqrt{3}\,k^3}{3} + \frac{(\sqrt{3} - 1)}{4}\,k^4\right]_{=0}^{=1}$$

$$= -\frac{wl^2}{24}\,(3 - \sqrt{3}) = -\cdot 053\,wl^2.$$

This is the bending moment caused by loading one whole centre span.

Referring to Plate I and thence to Fig. 93 we see that the arrangement (1) will give the greatest possible negative moment at point d.

Arrangement (2) gives M_d greatest possible positive value.

Arrangement (3) is for uniform load throughout.

Arrangement (4) gives the greatest positive moment at mid span h and also for points C and D on Plate I.

Thus in arrangement No. 1 in Fig. 93 we have

$$M_d = -\frac{wl^2}{24}\,(3 - \sqrt{3}) \times 2\left[1 + \frac{1}{(2 + \sqrt{3})^2} + \frac{1}{(2 + \sqrt{3})^4} \cdots\right]$$

$$= -\frac{wl^2}{24}\,(\sqrt{3} + 1) = -\cdot 114\,wl^2.$$

For arrangement (2)

$$M_d = +\frac{wl^2}{24}\,(\sqrt{3} - 1) = +\cdot 0305\,wl^2.$$

Arrangement (3) being a combination of (1) and (2) gives

$$M_d = \frac{wl^2}{24}\,(-\sqrt{3} - 1 + \sqrt{3} - 1) = -\frac{wl^2}{12}$$

which is obviously correct.

Arrangement (4) gives

$$M_d = M_e = -\frac{wl^2}{24}$$

$\therefore\ M_h$ (maximum positive value) $= +\frac{wl^2}{12} = +\cdot 083\,wl^2.$

Arrangement (5) gives

$$M_h = M_d = M_e = -\frac{wl^2}{24} = -\cdot 0417\,wl^2.$$

Points A and B on Plate I offer some difficulty for in each case the influence line changes value on one span.

We must therefore, in order to obtain the maximum and minimum values for these points, load only part of this span in each case.

We must first of all determine where the influence lines cut the span then determine by integration the bending moment caused by loading this part of the span to which we must add the moment due to loading other spans.

The author gives one case only, for point 1 (A on Plate I).

Let Fig. 86 represent a centre span.

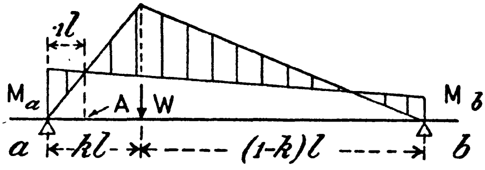

Fig. 86.

We wish to find the value of kl which will make the bending moment at $A = 0$.

Neglecting the signs of the quantities we have at A

$$+\,ve \text{ Moment} = \frac{W\,(1-k)\,l}{10}$$

$$-\,ve = M_a + \frac{M_b - M_a}{10} = \frac{9}{10}M_a + \frac{1}{10}M_b$$

$$\therefore\ \frac{W\,(1-k)\,l}{10} = \frac{9}{10}M_a + \frac{1}{10}M_b$$

$$W\,(1-k)\,l = 9\,M_a + M_b$$

$$= 9\,\frac{W\,l}{2}\,k\,(1-k)\,(1+(1-\sqrt{3})\,k)$$

$$+\ \frac{W\,l}{2}\,k\,(1-k)\,(1+(1+\sqrt{3})\,k)\,(2-\sqrt{3})$$

$$\frac{2W}{k} = 9\,W\,(1+(1-\sqrt{3})k) + W\,(2-\sqrt{3})\,(1+(1.+\sqrt{3})\,k)$$

i.e., $9\,k + 9\,k^2 - 9\sqrt{3}\,k^2 + 2\,k - \sqrt{3}\,k + \sqrt{3}\,k^2 - k^2 - 2 = 0$

$$k = \frac{+\,(11-\sqrt{3}) \pm \sqrt{(11-\sqrt{3})^2 - 8\,(8\sqrt{3}-8)}}{2\,(8\sqrt{3}-8)} = \cdot 256.$$

The values of M_a and M_b due to loading the span ab from a to $k = \cdot 256$, are, from our integral for the whole span,

$$M_a = \frac{w\,l^2}{2}\left[\frac{k^2}{2} - \frac{k^3}{\sqrt{3}} + (\sqrt{3}-1)\,\frac{k^4}{4}\right]_{=\,0}^{=\,\cdot 256} = -\ \cdot 012\,w\,l^2$$

$$M_b = \frac{w\,l^2}{2}\left[\frac{k^2}{2}+\frac{k^3}{\sqrt{3}}-(\sqrt{3}+1)\frac{k^4}{4}\right]_{\cdot 0}^{\cdot 256} \times (2-\sqrt{3}) = -\cdot 0056\,w\,l^2.$$

From which it follows that the actual bending moment at point A caused by loading from a to $k = \cdot 256$, $= +\cdot 0062\,w\,l^2$.

Maximum moment due to loading other spans $= +\cdot 0212\,w\,l^2$.

∴ Greatest positive value of bending moment at A

$$= +\cdot 0062 + \cdot 0212 = +\cdot 0274\,w\,l^2.$$

The values for the mid span envelope are shown on Table CB V.

The method of calculating the envelope for the end span is to apply an imaginary moment at the free end to reduce the conditions to those for a centre span. As the author thinks that he has already sufficiently illustrated this method the analysis will not be given.

Shears in a Continuous Beam.—If we know the loading on a span and the bending moments then we can by taking moments arrive at the shears.

Thus in Fig. 87 a centre span

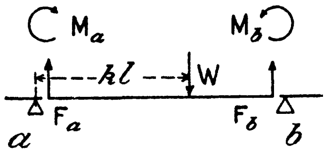

Fig. 87

we have
$$M_a - M_b + Wkl = F_b\,l$$
$$F_b = W\,k + \frac{M_a - M_b}{l}.$$

Putting in the values for M_a and M_b

$$F_b = W\,k - \frac{W\,k}{2}(1-k)(1-2k)(\sqrt{3}-1).$$

Putting $k = \cdot 1$, $= \cdot 2$, etc., we arrive at the values shown in Table CB VI.

The end span can be treated in a similar manner giving the values shown in Table CB VI.

Practical Application.—Bending Moment.—Comparing an actual reinforced concrete T-beam with the conditions specified for our ideal case we see at once that none of the necessary properties of the elastic beam are fully realized in the actual beam. Taking them seriatim as set out in the beginning of this chapter:—

(1) No practical beam has an infinite number of spans although beams with four or more equal spans approximate for bending moment very closely to this condition.

(2) The conditions at the supports will vary with actual cases. Secondary beams framing into main beams will be " restrained " by the torsional stiffness of the main beams. Main beams framing into columns will be restrained by the stiffness of the column. All practical supports sink under load. In the case of a long span main beam carrying short span secondaries this effect is large. For example suppose arrangement (1) in Fig. 93 represented such a case. Then support *d* at which the maximum moment occurs would sink below *c* and *e* and the moment at *d* would diminish. The greatest difference occurs in the breadth of the actual and assumed supports. Not only do the actual supports themselves possess a definite width but their effect is increased by the depth of the beam. In the author's opinion the " effective width " of any support may be taken as the actual width of the column plus the depth of the beam. This effective width averages about one-tenth of the span. (Compare the " column capital " Chapter XXIV.)

(3) *EI* varies in all T-beams and is assumed to change suddenly when the bending-moment changes sign throwing the concrete flange into tension when it becomes non-effective.

(4) The main effect of using a beam whose depth is about one-twelfth of the span has already been discussed in paragraph 2 above.

(5) The slope at any point of a beam is very small. On the other hand the occurrence of shear cracks seems to show that the deflection of an actual beam may be due partly to distortion of the web and not entirely to bending effect.

(6) The self-weight of a beam can be treated as an applied load and thus presents no difficulty. The initial stresses which exist in all structures should be covered by our assumed working stresses.

(7) The effect of this is not nearly so large as might be imagined, see Chapter XVI.

(8) The deflections are so small that end thrusts cannot affect the bending moments due to transverse loads although they may, if eccentric, produce additional moments of their own.

Taking all these considerations into account it would appear that :

(I) The " peak " values for reversed moment at the supports are never realized in an actual beam.

(II) The central moments are somewhat larger than those calculated.

(III) The effect of loading adjacent spans is smaller than calculated. The bending moments in actual cases are not very much different for internal and external panels and the provision of some constant bending-moment factor, irrespective of the number of spans (provided there are three or more) and the percentage of live load, is indicated.

The case of two equal spans may be treated as an exception.

Practical Application.—Shear.—The shear on all beams may be calculated as for a simply-supported span and in consequence the columns may be taken as carrying a series of simply-supported beams as far as direct load is concerned. This, of course, does not apply to cantilever construction.

Conclusion. Equal Spans.—Table CB VII gives the bending moment coefficients for continuous beams under combined self-weight and superload. These values are sanctioned by general usage and appear to the author to be founded on a very sound consideration of all aspects of the subject. Table CB VIII shows them graphically.

The extension to Tables CB III and CB IV is left to the reader.

A succession of rolling loads should be referred to the influence-lines on Plates I and II. In making the final design for the beam the designer should always remember that these influence lines are calculated for one certain ideal case. In arranging the bars, etc., the mid-span moments must be well covered while " the next size smaller " will do for the points of support.

Some designers when faced with the problem of a moving load system find, first of all, the maximum bending moment the system would produce on a freely-supported beam of the same span. They then take a fraction of this (usually $\frac{4}{6}$ for interior and $\frac{5}{6}$ for exterior spans). The bending moment at the support is then either guessed, or left to look after itself, some standard method of bending up the bars being used.

TABLE CB VII.

MAXIMUM BENDING MOMENTS (3 OR MORE EQUAL SPANS)
w = combined live and dead load

Interior Span	Interior Support	End Span	Next-to-end Support
$+\frac{wl^2}{12}$	$-\frac{wl^2}{12}$	$+\frac{wl^2}{10}$	$-\frac{wl^2}{10}$

TWO EQUAL SPANS

Span	Centre Support	Centre Reaction
$+\frac{wl^2}{10}$	$-\frac{wl^2}{9}$	$1{\cdot}20\,wl$

Apart from the uncertainty of the values so obtained, Plates I and II when drawn out for office use will give smaller values than most designers guess, the reason being the concave shape of the lines for the mid-span points. This fact alone should commend them to those whose object is economy of cost.

Unequal Spans.—This case closely resembles the case of equal spans with the additional complication that the moment of inertia may be much larger for one span than for another. It is not possible to run out influence lines for all possible cases of unequal spans and as a rule beams carrying point-load systems (viaducts, crane gantries, etc.) are made up of equal spans. The author has worked out two cases each having five spans, the centre span being of different length from the others.

Case I. Five spans.—*EI* constant.—Let Fig. 88 represent a system such that *ab*, *bc*, *de* and *ef* are equal spans $= l$ while *cd* is an odd span $= kl$.

The ends *a* and *f* are free.

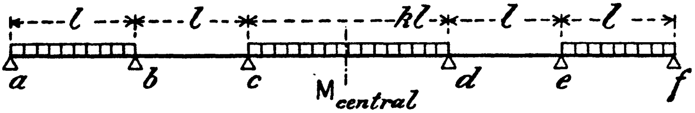

Fig. 88.

From Plate I it appears that the loading shown will produce a maximum value of the mid-span moment in the odd span.

w is the distributed load and is assumed to be the same for all spans.

By the theorem of three moments

$$4\,M_b + M_c = \tfrac{1}{4}\,wl^2$$

also
$$M_b\,l + 2\,M_c\,(l + kl) + M_d\,kl = \tfrac{1}{4}\,wk^3\,l^3$$

i.e.,
$$M_b + M_c\,(2\,l + 2\,kl + kl) = \tfrac{1}{4}\,wk^3\,l^3$$

$$M_c = wl^2 \times \frac{[k^3 - \frac{1}{4}]}{7 + 12\,k}\ \text{(numerically).}$$

(The usual statement of the Theorem of Three Moments gives the reversed moments as positive. In order to follow the general practice the author has done the same. In the final tabulation the direction of the moments is given on every curve.)

By putting
$k = \cdot 1$
$k = \cdot 2$
$k = 1{\cdot}1$
$k = 1{\cdot}2$, etc.,

we have a value of M_c in terms of wl^2.

As the loading is symmetrical we can then soon find the value of the mid span moment $M_{central}$.

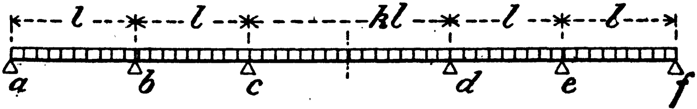

Fig. 89.

In order to estimate the dead load the case in Fig. 89 has also been taken. Applying the Theorem of Three Moments we have :—

$$M_c = wl^2 \times \frac{[k^3 + \frac{1}{2}]}{7 + 12k}.$$

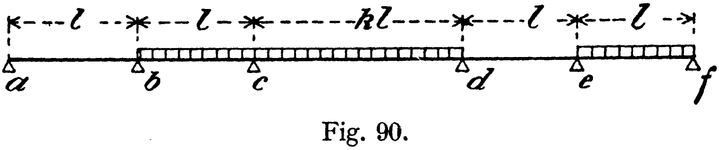

Fig. 90.

Fig. 90 gives us the maximum value of M_c

By analysis (not given here)

$$M_c\,[1{\cdot}75^2 + 7k + 3\,k^2] = \frac{wl^2}{16}\,[3 \times 1{\cdot}75 + 7\,k + 7\,k^3 + 4\,k^4].$$

Lemma IV.—Theorem of Three Moments (I varies).

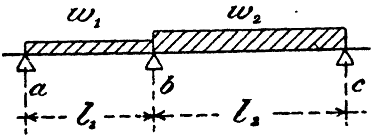

Fig. 91.

Suppose we have two adjacent spans *ab* and *bc* Fig. 91 carrying distributed loads w_1 and w_2 and assume $ab = l_1$, $bc = l_2$.

Imagine that the breadth *b* of the beam remains constant throughout while the depth varies as the span. Suppose for example that the depth of *ab* is $\frac{1}{20}$ of the span l_1 while the depth of *bc* is $\frac{1}{20}$ of the span l_2.

Then I_1 for $ab = Rl_1^3$

I_2 for $bc = Rl_2^3$, where R is a constant.

It can be shown that the slope at b* $= i_b$ is given by

$$EI_2\, i_b = \frac{w_2\, l_2^3}{24} - \frac{M_b\, l_2}{2} - \frac{(M_c - M_b)}{6}\, l_2$$

$$ER\, i_b = \frac{w_2}{24} - \frac{M_b}{2\, l_2^2} - \frac{(M_c - M_b)}{6\, l_2^2}$$

$$6\ ER\, i_b = \frac{w_2}{4} - \frac{2M_b}{l_2^2} - \frac{M_c}{l_2^2}.$$

Similarly for span *bc*

$$-\,6\ ER\, i_b = \frac{w_1}{4} - \frac{2\,M_b}{l_1^2} - \frac{M_a}{l_1^2}$$

---

*See Morley's *Strength of Materials.*

$$\therefore \quad \frac{M_a}{l_1^2} + 2\,M_b\left(\frac{1}{l_1^2}+\frac{1}{l_2^2}\right) + \frac{M_c}{l_2^2} = \left(\frac{w_1}{4}+\frac{w_2}{4}\right).$$

If as before we write $l_2 = kl$
$l_1 = l$

we have $M_a + 2M_b\left(1 + \frac{1}{k^2}\right) + \frac{M_c}{k^2} = \left(\frac{w_1}{4} + \frac{w_2}{4}\right) l^2.$

If $w_1 = w_2$ as we assume, this $= \frac{wl^2}{2}$.

Case II.—I varies as span³.

Adopting a loading similar to Fig. 88

we have :— $M_a + M_b \times 4 + M_c = \frac{1}{4} wl^2$

$$M_b + 2\,M_c\left[1 + \frac{1}{k^2}\right] + \frac{1}{k^2} \times M_d = \frac{1}{4} wl^2$$

$$M_c = M_d$$

giving $M_c = \frac{3}{16} \times \frac{wl^2}{\left[1{\cdot}75 + \frac{3}{k^2}\right]}.$

Loadings similar to Fig. 89 and Fig. 90 have also been worked out. As there is little of interest in the analysis it is omitted.

Table CB IX.—This table shows the results of the previous analysis. The bending moments are shown as decimals of the larger span in all cases. That is, when k is less than unity the bending moments are in terms of wl^2 : when k is greater than unity they are shown in terms of $w\ (k\,l)^2$. It will be seen that the upper curve shows a marked difference from the lower one. When the moment of inertia varies, the span tends to behave more after the fashion of an isolated span freely supported. These curves have been drawn for values of k ranging from 0 to 3. For higher values the ultimate limit of the bending moments is readily visible. The upper curve when k tends to infinity becomes a single span with free ends, while the lower curve becomes a single span with fixed ends. Which case will be followed in practice depends entirely on the particular application. If the example is not important (say a few bays of floor slab having one bay smaller than the rest) the same section will be carried right through and EI may be taken as constant. If the example is large (say a bridge with several small approach spans and one large span in the centre) then the section will undoubtedly be varied and EI will change.

The author knows of no sound method for arriving quickly at the moments in unequal spans for all cases*. The reader should work out several cases similar to Table CB IX and tabulate them. Plates I and II indicate the distribution of load which will give the maximum moments. The values for EI = constant are obtained by the Theorem of Three Moments while the values for EI = constant × span^3 are obtained by the modified theorem given in the analysis under Lemma IV.

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* The author's own ideas on this matter are too revolutionary to be included in a text-book.

Any one particular case, can, of course, be worked out by itself but this takes time and reference to tabulated values which approximate closely to the case in point may prove quicker.

The designer must never forget that an actual beam will differ from his assumed conditions of supporting and loading, and must make allowance accordingly.

Also it must be remembered that the number of bars in an actual beam is small and the arrangement of them admits of little variation. The questions of which moments should be studied first, and which moments determine the section of beam to be used are inextricably associated with the possible arrangement of bars in the beam and the reader will find further discussion in the chapters devoted to beams.

Span.—In all continuous bays the span centre to centre of supports must be taken (not the clear span). The lengths of end spans and the effect of wide supporting beams are discussed later on.

The idea of providing a separate table for beams loaded at the third points is taken from *Reinforced Concrete Design* by Oscar Faber, Vol. II. The values given are from the author's own influence lines.

EXAMPLES ON CHAPTER X.

No examples will be given here as sufficient cases occur later on. The reader must work out at least one or two cases similar to the following.

A continuous beam has four spans. The two left-hand spans = l while the two right hand spans = $k\,l$. Give the maximum, minimum and " dead load " bending moments at the centre support for values of k from 0 to 3 in a form similar to CB IX.

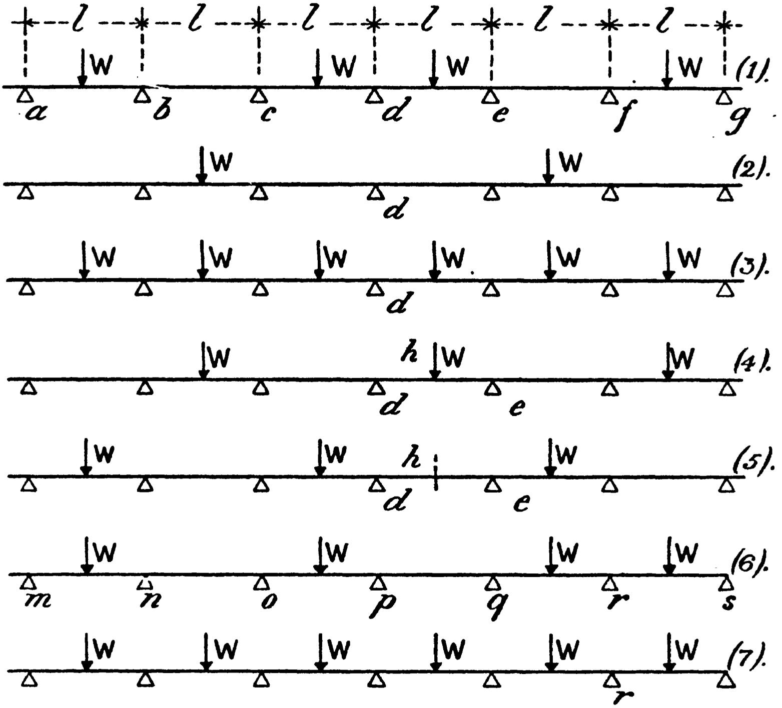


Fig. 92

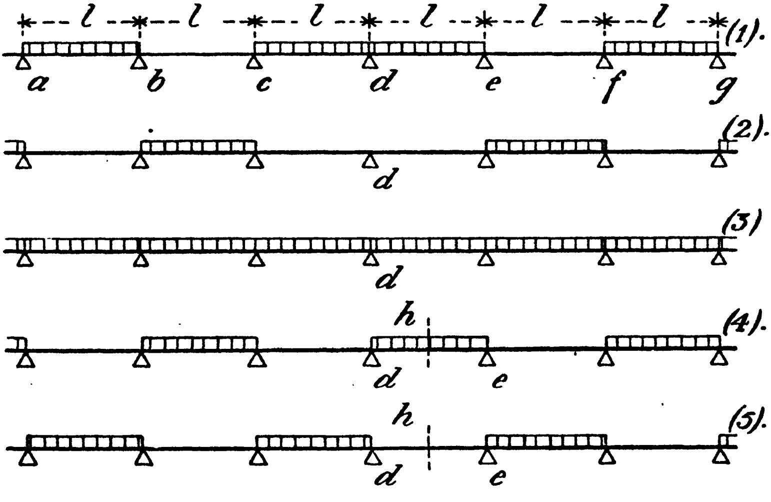


Fig. 93.

TABLE CB I (CENTRE SPANS)

Rolling Load W on Point	BENDING MOMENTS IN TERMS OF $W L$ AT POINTS										
	Support	1	2	3	4	Mid Span	6	7	8	9	Support
1	— ·0419	+ ·0508	+ ·0434	+ ·0360	+ ·0287	+ ·0214	+ ·0141	+ ·0067	— ·0006	— ·0080	— ·0153
2	— ·0682	+ ·0153	+ ·0988	+ ·0823	+ ·0658	+ ·0494	+ ·0328	+ ·0164	— ·0001	— ·0166	— ·0331
3	— ·0820	— ·0089	+ ·0642	+ ·1373	+ ·1104	+ ·0835	+ ·0565	+ ·0296	+ ·0027	— ·0242	— ·0511
4	— ·0850	— ·0233	+ ·0385	+ ·1002	+ ·1619	+ ·1237	+ ·0854	+ ·0471	+ ·0088	— ·0294	— ·0677
Mid	— ·0791	— ·0291	+ ·0209	+ ·0709	+ ·1209	+ ·1709	+ ·1209	+ ·0709	+ ·0209	— ·0291	— ·0791
6	— ·0677	— ·0294	+ ·0088	+ ·0471	+ ·0854	+ ·1237	+ ·1619	+ ·1002	+ ·0385	— ·0233	— ·0850
7	— ·0511	— ·0242	+ ·0027	+ ·0296	+ ·0565	+ ·0835	+ ·1104	+ ·1373	+ ·0642	— ·0089	— ·0820
8	— ·0331	— ·0166	— ·0001	+ ·0164	+ ·0328	+ ·0494	+ ·0658	+ ·0823	+ ·0988	+ ·0153	— ·0682
9	— ·0153	— ·0080	— ·0006	+ ·0067	+ ·0141	+ ·0214	+ ·0287	+ ·0360	+ ·0434	+ ·0508	— ·0419

TABLE CB II (END SPAN)

Rolling Load W on Point	BENDING MOMENTS AT POINTS IN TERMS OF $W L$										
	Support	1	2	3	4	Mid Span	6	7	8	9	Support
1	— ·0460	+ ·0486	+ ·0432	+ ·0378	+ ·0324	+ ·0270	+ ·0216	+ ·0162	+ ·0108	+ ·0054	0
2	— ·0770	+ ·0107	+ ·0984	+ ·0861	+ ·0738	+ ·0615	+ ·0492	+ ·0369	+ ·0246	+ ·0123	0
3	— ·0950	— ·0155	+ ·0640	+ ·1435	+ ·1230	+ ·1005	+ ·0820	+ ·0615	+ ·0410	+ ·0205	0
4	— ·1040	— ·0345	+ ·0360	+ ·1065	+ ·1770	+ ·1475	+ ·1180	+ ·0885	+ ·0590	+ ·0295	0
Mid	— ·1020	— ·0400	+ ·0200	+ ·0800	+ ·1400	+ ·2000	+ ·1600	+ ·1200	+ ·0800	+ ·0400	0
6	— ·0900	— ·0410	+ ·0080	+ ·0570	+ ·1060	+ ·1550	+ ·2040	+ ·1530	+ ·1020	+ ·0510	0
7	— ·0730	— ·0375	0	+ ·0375	+ ·0750	+ ·1125	+ ·1500	+ ·1875	+ ·1250	+ ·0625	0
8	— ·0520	— ·0268	— ·0016	+ ·0236	+ ·0486	+ ·0740	+ ·0992	+ ·1244	+ ·1496	+ ·0748	0
9	— ·0270	— ·0139	— ·0012	+ ·0114	+ ·0241	+ ·0367	+ ·0494	+ ·0620	+ ·0746	+ ·0873	0

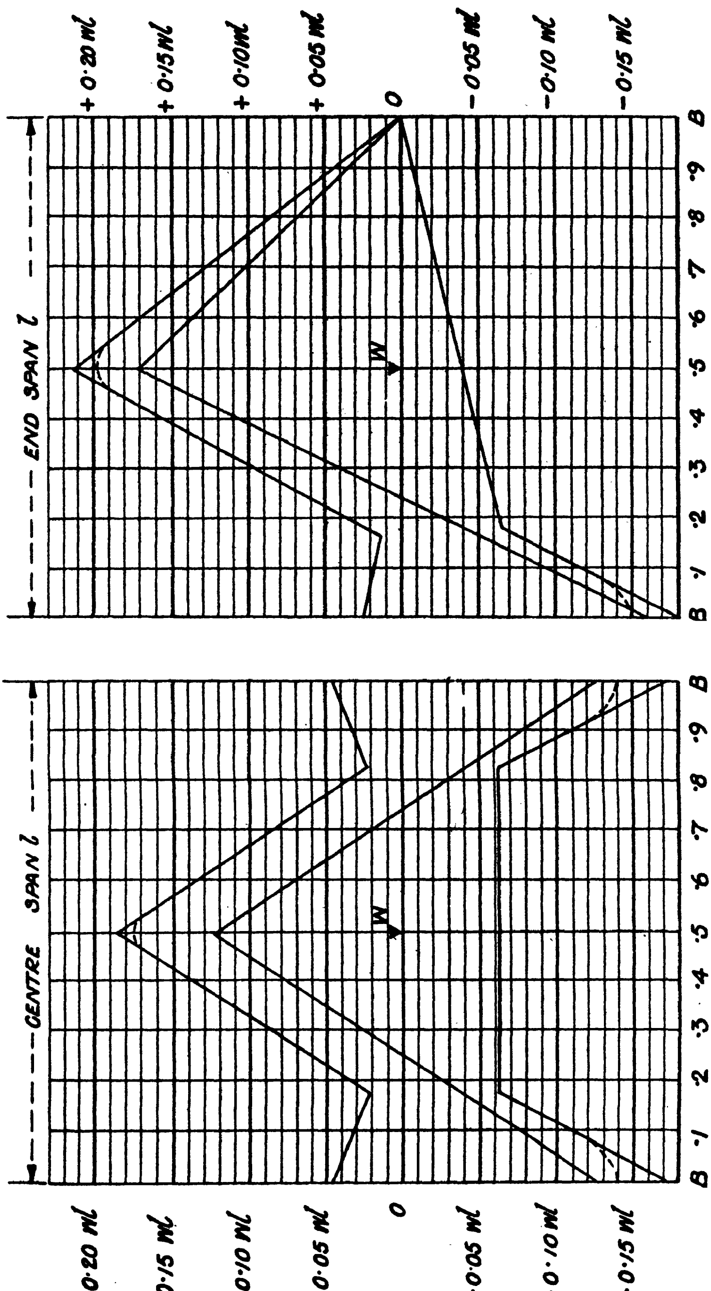


TABLE C.B.III ENVELOPE OF BENDING MOMENTS INFINITE NUMBER OF SPANS

CENTRE POINT LOAD
E I = CONSTANT

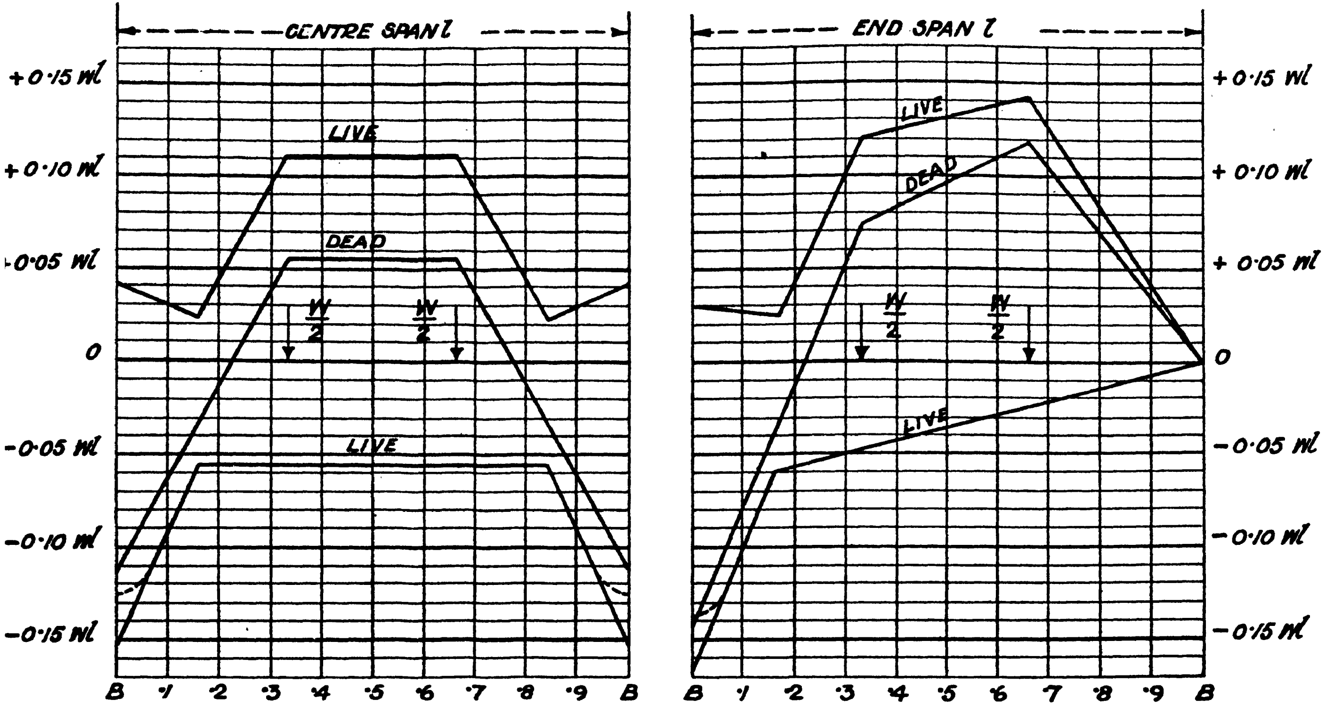


TABLE C.B IV. Envelope of Bending Moments loads at Third Points
Infinite Number of Spans EI = Constant

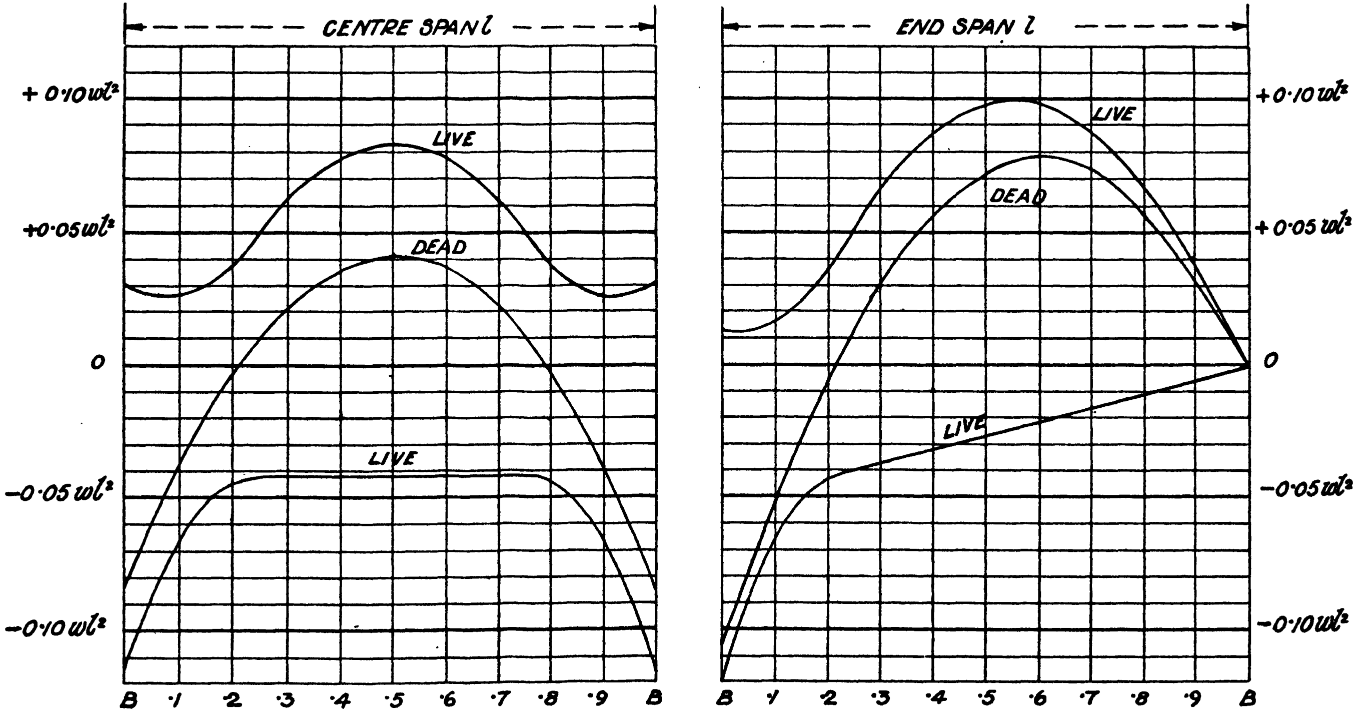


TABLE C.B. V ENVELOPE OF BENDING MOMENTS FOR SUPER LOAD w
INFINITE NUMBER OF SPANS EI = CONSTANT

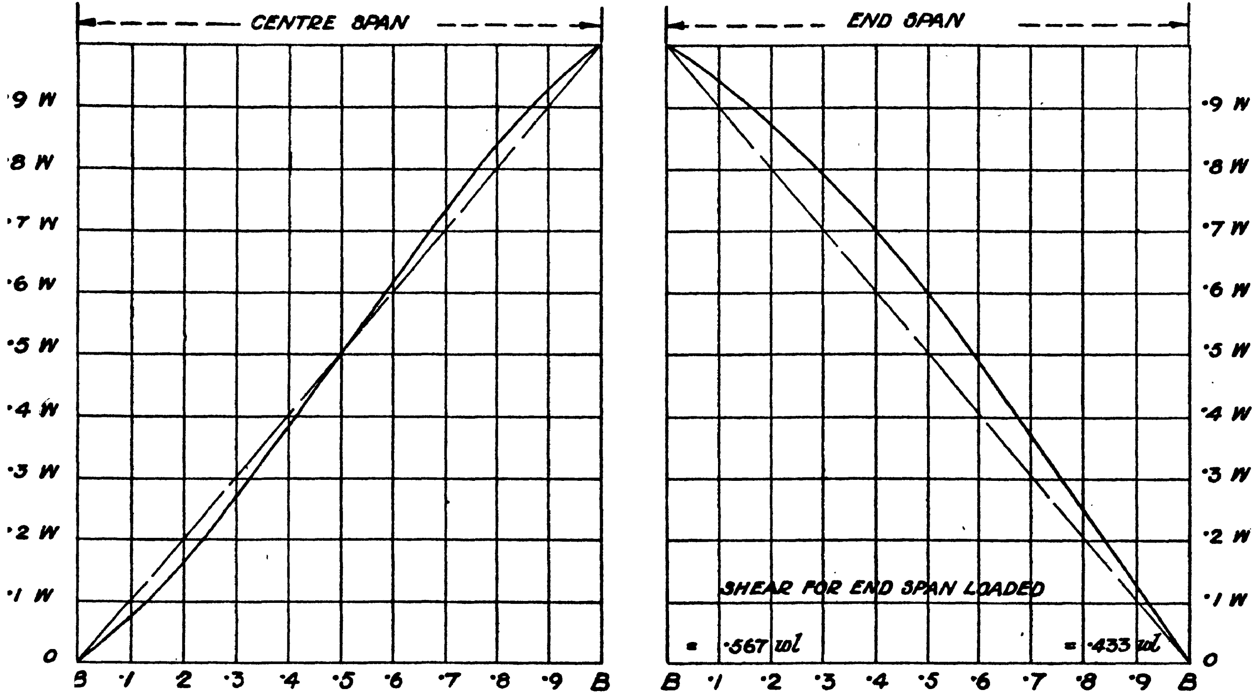


TABLE C.B.VI INFLUENCE LINES FOR SHEAR FOR SINGLE ROLLING LOAD W
INFINITE NUMBER OF SPANS EI = CONSTANT

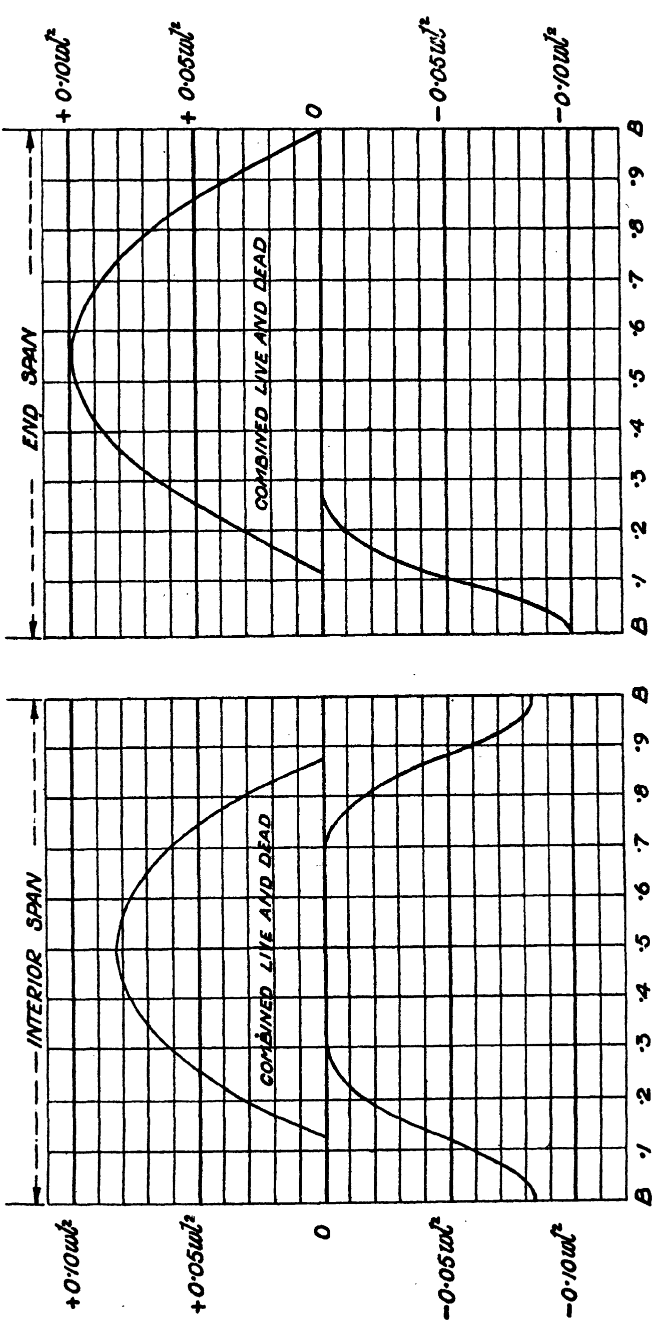


TABLE C.B. VIII

PROBABLE ENVELOPE OF BENDING MOMENTS IN ACTUAL BEAM
(3 OR MORE EQUAL SPANS)

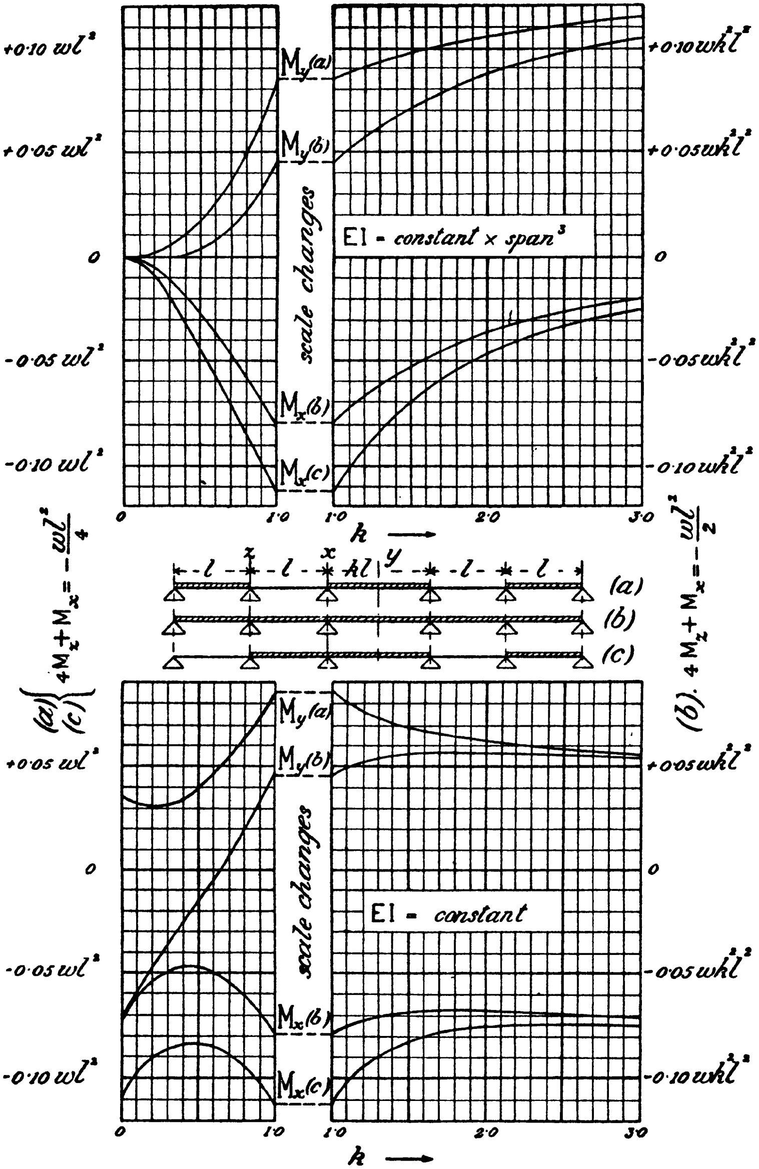


TABLE C.B.IX. CONTINUOUS BEAMS UNEQUAL SPANS
(BEAMS CONVEX UNDERNEATH HAVE POSITIVE MOMENT).

The above table gives the Bending Moments at points x and y for the three loadings shewn. The upper curve is for the Modified Theorem of Three Moments.

CHAPTER XI

REINFORCEMENT—QUANTITIES

Material.—Some few designers have advocated and used high tensile steel as reinforcement. Both high-carbon steel and steel whose yield point has been raised by being twisted or deformed cold, have been employed. The difficulties of bending and placing such material will in most cases far outweigh the saving in weight of steel (which saving, as a rule, is offset by increased price). Expanded metal or some kind of wire-mesh may be used in road foundations, " plain " concrete floors laid directly on the ground, partition-walls, etc., although the use of any steel with sharp or jagged edges should be avoided as it will cut the hands of the men who place it, especially in cold weather when the men's hands get numb. In such cases where the steel is put in to prevent shrinkage and temperature cracks the engineer should always work out the price per ton of the particular mesh he intends to employ and compare it with the current price of mild steel bars per ton. Considering the amount of cutting and bending, it is not surprising that mild steel bars are used as reinforcement in almost every case. It is obviously difficult to give exact figures but the author opines that at least 98% of modern reinforcement is mild steel.

Links and Stirrups.—These constitute the secondary reinforcement and are placed in a plane at right angles to the axis of the member. They have therefore to lie within the minimum cross section and are comparatively short. In an average beam or column the height of the stirrups or links will be only $\frac{1}{20}$th to $\frac{1}{30}$th of the length of the main bars. Moreover the secondary steel has to be bent round the main steel which means that all the links in a column, for example, have to be bent exactly to the same size, as otherwise they would not all fit tight round the main bars. Secondary steel must be easily bent and must in consequence be either small bars or thin flats.

The main objection to the employment of thin flats is their width as flat stirrups closely spaced leave little room for concrete to pass between them thus creating a plane of weakness. Another drawback is that a flat stirrup which has accidentally been placed too near an exposed surface is more liable to rust and blow off the concrete cover than a round bar similarly placed. Against this it must be stated that a flat stirrup is, area for area, more flexible than a round bar. In the case of links or stirrups whose two extremities are hooked round the same main bar, flat sections are unsuitable. In all the designs in this work round bars will be used as stirrups and links. Should the reader wish to use flat section stirrups he can easily do so, the methods of calculating the sectional areas required can be applied to all sections.

The diameters usually employed as links or stirrups are $\frac{3}{16}$-in. diameter to $\frac{3}{8}$-in. diameter for ordinary floor beams, and of these sizes most designers prefer to use $\frac{1}{4}$-in. diameter and $\frac{3}{8}$-in. diameter only. As a very rough guide the diameter of stirrups should not exceed $\frac{1}{40}$ of their depth. Thus it would be unusual to employ $\frac{1}{2}$-in. diameter stirrups in a beam less than 24 ins. overall depth or $\frac{5}{8}$-in. diameter stirrups in a beam less than 30 ins. overall depth. In one case which came under the author's notice, $\frac{3}{4}$-in. diameter stirrups were used throughout, the overall depth of the beams ranging from 36 ins. to 96 ins. This is, of course, exceptionally heavy construction.

End hooks on $\frac{1}{4}$-in. links should be about $2\frac{1}{2}$ ins. long and on $\frac{3}{8}$-in. links about 3 ins. long. These details are, of course, more a question of construction than of design, but it is essential when detailing members to keep the diameter of the links and stirrups as small as can be reasonably arranged.

Main Bars.—The bars in floor and wall slabs call for little comment. The main bars in a floor are usually not less than $\frac{3}{8}$-in. diameter while the cross-rods are not less than $\frac{1}{4}$-in. diameter. It is always as well to remember that $\frac{5}{8}$-in. diameter bars are charged at basis rate while smaller sizes are charged extra. Further discussion of the size and spacing of bars in slabs is given in the chapter dealing with those members.

The bars employed as main bars in columns range from $\frac{1}{2}$-in. diameter to $1\frac{1}{2}$ in. diameter.

The main bars in beams range from $\frac{1}{2}$-in. diameter to $1\frac{1}{8}$ in. diameter for usual construction and up to $1\frac{1}{2}$ in. for heavy construction. Bars up to $1\frac{1}{8}$ in. diameter may be bent in a single-lever bending-machine with two men but larger sizes require a geared-machine or a specially heavy long lever-machine. Bars up to 1 in. diameter can be cut on a light single lever shears, but large sizes require a heavier machine or a hack-saw or an oxy-acetylene flame.

Very large rods 2 ins. to 3 ins. diameter are sometimes used in tension members. Such large rods should be passed through mild-steel anchor plates and held with a nut and back-nut. (If the full diameter of the bar is required right up to the end, the end must be upset before the thread is cut.)

Bars up to $1\frac{1}{2}$ in. diameter can readily be obtained up to 40 ft. 0 in. in length. The reader should refer to the rolling mills for the maximum lengths and weights that can be rolled and transported and the extras which are charged.

Adhesion of Bars. Grip Length.—If a plain bar of diameter d is embedded in a block of concrete, such that a length l is buried in the block as shown in Fig. 94 the concrete will adhere to the bar and a considerable tension T is required to pull it out of the block. It is found that this tension may best be expressed as a frictional force per square inch of embedded surface of the bar. In the case illustrated the perimeter of the bar is πd giving an area of surface of $\pi d l$.

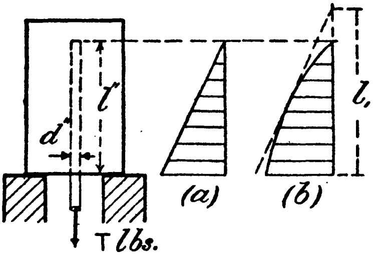


Fig. 94.

If the force necessary to pull out the bar is T lbs. then the ultimate adhesion per square inch

$$= \frac{T}{\pi d l}.$$

It is found by experiment that a safe adhesive force of 100 lbs. per square inch may be relied on for average 1 : 2 : 4 concrete.

If the bar is fully stressed at the place where it leaves the block to 18,000 lbs. per sq. in.

then $$T = 18{,}000 \times \frac{\pi d^2}{4} = 100 \, \pi d l$$

giving $l = 45\, d$

i.e., embedded length = 45 diameters for a plain round bar.

Flat section bars have a larger perimeter per unit area of section but do not as a rule realize the full calculated value of the adhesion if the perimeter is very large compared with the area. In the case of flat stirrups for example, 1 in. × ⅛-in. to 2 ins. × ¼-in. section, about 66 lbs. per sq. in. may be taken.

There are on the market a great number of deformed and indented bars which give higher values than 100 lbs. per sq. in. A careful arrangement of plain bars will allow ample grip length in all members which renders the use of indented sections unnecessary.

End Anchors.—As mentioned above, very large bars should have anchor-plates. In modern first-class work all main tension bars are bent or deformed at the extremities with the object of increasing the safe tensile stress near the ends. With the high-shrinking cement at present in use the author uses end-hooks on practically all bars except column bars. The most common forms of end anchorage are shown in Fig. 95 namely (*a*) round hook, (*b*) square hook, (*c*) fish-tail. The slight flattening and perfunctory notch which is sometimes provided as a " fish-tailed end " are of no use whatever.

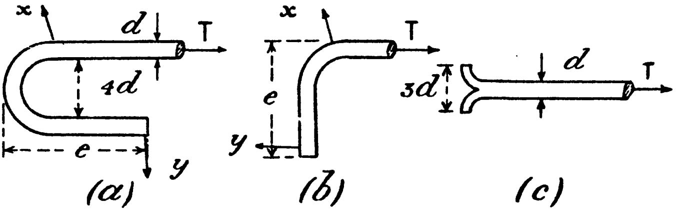


Fig. 95.

It will be seen that all these hooks tend to straighten out when the bar is in tension and that forms (*a*) and (*b*) are not symmetrical. In cases where the hook is well buried in concrete at all points, form (*a*) will provide the best grip. If, however, either the spare end or the back of the hook is near the surface the bar will tend to straighten out, the spare end tending to move in the direction *y* as shown and the back of the hook in the direction *x*. In cases where it is essential to develop the full strength of the bar in a very short length the end should be bent down into the centre of the beam or stirrups arranged specially to hold in the hook. Such cases are not sufficiently frequent to justify the general use of indented or deformed bars. The length marked *e* in Fig. 95 is about ten bar-diameters for the round hook (*a*) say 4½ ins. for a ½-in. bar and 1 ft. 0 ins. for a 1⅛ in. bar. For large bars it may be cheaper to bend the end-hooks hot but no practical designer would allow this as the presence of a forge near the bar-benders' bench must inevitably lead to misuse of same. In any case all intermediate bends must be bent cold. The round hook in type (*a*) has now become the most popular type of end anchorage.

Bond Stress in Beam Bars.—The theoretical shear-bond stress at a given section is given in Chapter IX and elaborated in Chapter XIV. The case of a bar in a beam is not exactly similar to the case shown in Fig. 94 as the concrete in this latter case is under compression while the bar is in tension. Standard practice and tests show that a grip length of 45 diameters is safe in both cases for a stress of 18,000 lbs. per sq. in. in average 1 : 2 : 4 concrete provided the increase in stress is steady.

Referring again to Fig. 94 we see that the bar carries a stress of 18,000 lbs. per sq. in. at its lower end where it leaves the concrete. At its upper end the stress is zero and is assumed to vary uniformly from top to bottom in direct proportion to the embedded length counted from the free end (i.e., the top end in the figure). The stress diagram (*a*) represents this case. If the stress in a bar increased as shown in stress diagram (*b*) it would be necessary to extend the bar to a length l_1 as drawn, to enable the upper end of the bar to pick up enough adhesion to withstand the stress near the top end where the increase of stress is much more rapid than in diagram (*a*). In a beam the stresses are, theoretically, determined by the radius of curvature to which the beam is bent and if a bar occurs at any section its stress must be equal to that of other neighbouring bars at that section.

Let Fig. 96 represent two sections of a beam $a - a$ and $c - c$ which lie close together, the bending moment being M. Suppose the

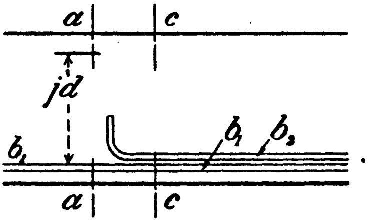


Fig. 96.

beam has one bar whose area is b_1 which runs through both sections and one bar b_2 which stops between the sections.

If $j\,d$ is the lever arm we have

$$\text{stress in bar } b_1 \text{ at } a - a = \frac{M}{j\,d \times b_1}$$

$$\text{stress in bars } b_1 \text{ and } b_2 \text{ at } c - c = \frac{M}{j\,d \times (b_1 + b_2)}.$$

Thus the stress in the bar b_2 has increased from zero to $\frac{M}{j\,d\,(b_1 + b_2)}$ between the two sections. By making the two sections approach one another it is easily shown that this stress must theoretically exist at the very end of bar b_2 provided that this bar does not move relatively to the surrounding concrete. We have therefore two alternatives :—

(1) to provide a really efficient end-hook

(2) to stop the bar at a point where the moment is zero.

The second of these two is the better course and should always be aimed at, the bars being stopped at point of minimum bending moment or being bent up away from the tensile flange past the neutral axis. The local stresses set up by anchoring bars at places of high stress are very heavy and may cause cracks which prove very serious in watertight structures.

If there are a lot of large bars requiring anchorage they should be stopped at different points so as to distribute these local stresses as evenly as possible.

It is usual to provide a lap of one quarter the span past the support (see the chapter on T-beams) which brings the end of the bar to a place of minimum bending moment (see Table CB VIII). If we specify 45 diameters grip length from the end of the bar to the point of maximum reversed moment which occurs at the support we have

$$45 \text{ diams.} = \tfrac{1}{4} \text{ span or less,}$$

$$\text{or diameter of bar} = \frac{\text{span}}{180} \text{ or less.}$$

This is a very rough rule but is useful as indicating the limit in usual cases. Thus in a beam 20 ft. 0 in. span the bars should not exceed

$$\frac{20 \times 12}{180} = 1{\cdot}33 \text{ say } 1\tfrac{1}{4}'' \text{ diameter.}$$

Further approximate relations between span and bar diameter are given in Chapter XIV.

Competitive Design Quantities.—The main object in carrying out a preliminary competitive design is to arrive at the quantities and thence to estimate the cost of the structure. The three main items of concrete, steel and centering are usually worked out at the time the design is made. Methods of stating unit quantities vary slightly, although the underlying principle is the same in all cases. Some offices work entirely in cubic yards and hundredweights of steel. Others in cubic feet and pounds. Some beginners aim at meticulous and laborious accuracy in taking off quantities. Such " accuracy " is entirely fictitious and a waste of time and of office costs. Quantities should be taken on the gross centre-line measurements, the only deductions being actual open spaces and sizable holes. The floor-slabs are taken over the whole floor area, the columns the whole floor height (top of floor to top of floor) and the beams on their gross overall length. This, of course, includes the concrete and centering at the intersections of floor and columns and at the intersection of main and secondary beams twice over. This fact is a matter of no importance. The man who makes the estimate understands the method of taking off that has been employed and his unit prices are based on such a system. Not only does centre-line measurement save time and cost in competitive design but it greatly facilitates the task of estimating the weight of steel, because the steel bars go through all intersections.

In this book the concrete will be stated in cubic yards, the steel in pounds, and the centering in square yards. The steel may be converted from pounds to hundredweights as the unit quantities are taken from the calculations and written in the quantity sheets.

Bills of Quantities.—In a measured job the quantities are taken off with some degree of accuracy before the job goes out to tender, as fairly complete preliminary drawings are available and more time may be spent on them. On the completion of the contract the volumes of concrete and areas of shuttering are calculated from the final detailed drawings and the exact weight of steel is obtained from the steel sheets. Such final measurement may take weeks and is an exercise in painstaking geometry not to be confused with the wild scramble of getting out preliminary quantities for a competitive design from a few half-finished small-scale sketches.

General Proportions.—Taking a large number of complete structures the ratio of steel to concrete ranges from 1 to 3 cwts. per cubic yard with an average of possibly 1¾ cwts. per cu. yd.

A few examples at random from recent contracts are given below, showing the weight of steel and area of shuttering per cubic yard of concrete.

Engine Foundation Blocks	1 cu. yd.	0·24 cwts.	1·7 sq. yds.
Flat-Slab Warehouse	1 cu. yd.	1·29 cwts.	4·32 sq. yds.
Office Building	1 cu. yd.	1·93 cwts.	7·10 sq. yds.
Water Tower	1 cu. yd.	2·3 cwts.	7·85 sq. yds.
Reservoir	1 cu. yd.	1·78 cwts.	5·50 sq. yds.
Coke Bunker	1 cu. yd.	1·63 cwts.	11·2 sq. yds.
Large Coal Bunker, etc.	1 cu. yd.	2·11 cwts.	6·85 sq. yds.
Silo Building	1 cu. yd.	1·42 cwts.	9·20 sq. yds.
Bridge	1 cu. yd.	2·98 cwts.	4·22 sq. yds.
Piled Jetty superstructure	1 cu. yd.	2·1 cwts.	5·95 sq. yds.
Swimming Bath and Buildings	1 cu. yd.	1·12 cwts.	6·82 sq. yds.

In all cases the figures are for the superstructures only. Deep foundations have been omitted from the figures. The ratio of shuttering to concrete is about 4 to 12 sq. yds. per cub. yd. with a possible average of 7 sq. yds. per cu. yd.

Some designers run out the quantities of concrete and shuttering and attempt to estimate the steel by guessing the average weight of steel per cu. yd. of concrete. This usually ends in disaster as the ratio of steel to concrete is by no means constant even in structures of a very definite type such as silos. The unit weight of steel in each member should always be calculated as this is the only reliable method.

In working out unit quantities of concrete no allowance is made for the volume occupied by the steel.

Tables B I and B II.—These tables show the sectional area of steel per foot of section which would appear if a section was drawn across a series of parallel bars spaced as shown. Table B I which is intended for stirrups is in ascending values of area.

The method of making such tables is simple, for example the sectional area of a $\frac{1}{2}$-in. round bar is ·1963 sq. ins. If $\frac{1}{2}$-in. bars were laid out in a row the bars being 4 ins. centre to centre and a section was taken at right angles to all the bars, then three sections of bar would appear to each foot run of section giving an area per foot of

$$3 \times \cdot 1963 = \cdot 59 \text{ sq. ins.}$$

It will be noted that all sizes up to 1 in. diam. are given for every $\frac{1}{16}$-in. This does not mean that all such sizes can be used in one design. The number of different sizes of bar used on one job must be as small as possible. On no account should large sizes, differing by $\frac{1}{16}$-in. only, be used on the same job.

If the designer has used $\frac{7}{8}$ in. bars and 1 in. bars in a design he should not use $\frac{15}{16}$-in. bars as well. If the steel bender were in a hurry (as he should be) he would not be able to distinguish between the three sizes, especially if the bars were rusty, and confusion and loss of time (and money) would result.

Table B II shows $1\frac{1}{2}$ in. bars spaced at 1 in. centres. This is done to include the value 21·2 sq. ins. which is the area of 12 bars $1\frac{1}{2}$ in. diam. and makes the table useful for designing beam sections having 12 tension bars, etc.

TABLE B I (ROUND BARS)

SECTIONAL AREA PER FOOT RUN OF SECTION FOR ROUNDS

DIAM. INCHES	SPACING INCHES	AREA SQ. INS.	DIAM. INCHES	SPACING INCHES	AREA SQ. INS.
$\frac{3}{16}$	12	·0276	$\frac{1}{4}$	5	·118
$\frac{3}{16}$	11	·0301	$\frac{3}{8}$	11	·121
$\frac{3}{16}$	10	·0331	$\frac{1}{4}$	$4\frac{1}{2}$	·131
$\frac{3}{16}$	9	·0368	$\frac{5}{16}$	7	·132
$\frac{3}{16}$	8	·0414	$\frac{3}{8}$	10	·133
$\frac{3}{16}$	7	·0473	$\frac{1}{4}$	4	·147
$\frac{1}{4}$	12	·0491	$\frac{3}{8}$	9	·147
$\frac{1}{4}$	11	·053	$\frac{5}{16}$	6	·153
$\frac{3}{16}$	6	·055	$\frac{3}{8}$	8	·165
$\frac{1}{4}$	10	·059	$\frac{5}{16}$	$5\frac{1}{2}$	·167
$\frac{3}{16}$	$5\frac{1}{2}$	·060	$\frac{1}{4}$	$3\frac{1}{2}$	·168
$\frac{1}{4}$	9	·065	$\frac{5}{16}$	5	·184
$\frac{3}{16}$	5	·066	$\frac{3}{8}$	7	·189
$\frac{1}{4}$	8	·074	$\frac{1}{4}$	3	·196
$\frac{3}{16}$	$4\frac{1}{2}$	·074	$\frac{5}{16}$	$4\frac{1}{2}$	·205
$\frac{5}{16}$	12	·077	$\frac{3}{8}$	6	·221
$\frac{3}{16}$	4	·083	$\frac{5}{16}$	4	·230
$\frac{5}{16}$	11	·084	$\frac{3}{8}$	$5\frac{1}{2}$	·241
$\frac{1}{4}$	7	·084	$\frac{5}{16}$	$3\frac{1}{2}$	·263
$\frac{5}{16}$	10	·092	$\frac{3}{8}$	5	·265
$\frac{3}{16}$	$3\frac{1}{2}$	·095	$\frac{3}{8}$	$4\frac{1}{2}$	·295
$\frac{1}{4}$	6	·098	$\frac{5}{16}$	3	·307
$\frac{5}{16}$	9	·102	$\frac{3}{8}$	4	·331
$\frac{1}{4}$	$5\frac{1}{2}$	·107	$\frac{3}{8}$	$3\frac{1}{2}$	·379
$\frac{3}{16}$	3	·110	$\frac{3}{8}$	3	·442
$\frac{3}{8}$	12	·110	$\frac{3}{8}$	$2\frac{1}{2}$	·530
$\frac{5}{16}$	8	·115	$\frac{3}{8}$	2	·662

TABLE B II (ROUND BARS)

DIAMETER INCHES	AREA SQUARE INCHES	WEIGHT PER FOOT LBS	SECTIONAL AREA OF STEEL PER FOOT RUN OF SECTION WHEN SPACED AT															
			1″	1½″	2″	2½″	3″	3½″	4″	4½″	5″	5½″	6″	7″	8″	9″	10″	12″
$\frac{1}{4}$	·0491	·167	·59	·39	·29	·23	·20	·17	·15	·13	·12	·11	·10	·08	·07	·07	·06	·05
$\frac{5}{16}$	·0767	·261	·92	·61	·46	·36	·31	·26	·23	·20	·18	·17	·15	·13	·11	·10	·09	·08
$\frac{3}{8}$	·1104	·376	1·32	·88	·66	·53	·44	·38	·33	·29	·26	·24	·22	·19	·17	·15	·13	·11
$\frac{7}{16}$	·1503	·511	1·81	1·21	·90	·72	·60	·51	·45	·40	·36	·33	·30	·26	·23	·20	·18	·15
$\frac{1}{2}$	·1963	·668	2·38	1·57	1·18	·94	·78	·67	·59	·52	·47	·43	·39	·34	·29	·26	·24	·20
$\frac{9}{16}$	·2485	·845	2·99	1·99	1·49	1·19	·99	·85	·75	·66	·60	·54	·50	·43	·37	·33	·30	·25
$\frac{5}{8}$	·3068	1·043	3·68	2·46	1·84	1·47	1·23	1·05	·92	·82	·74	·67	·61	·53	·46	·41	·37	·31
$\frac{11}{16}$	·3712	1·262	4·45	2·97	2·23	1·78	1·48	1·27	1·11	·99	·89	·81	·74	·64	·56	·49	·45	·37
$\frac{3}{4}$	·4418	1·502	5·30	3·54	2·65	2·12	1·77	1·51	1·32	1·18	1·06	·96	·88	·76	·66	·59	·53	·44
$\frac{13}{16}$	·5185	1·763	6·22	4·15	3·11	2·48	2·07	1·78	1·56	1·38	1·24	1·13	1·04	·89	·78	·69	·62	·52
$\frac{7}{8}$	·6013	2·044	7·23	4·81	3·61	2·88	2·40	2·06	1·80	1·60	1·44	1·31	1·20	1·03	·90	·80	·72	·60
$\frac{15}{16}$	·6903	2·347	8·30	5·54	4·14	3·31	2·76	2·37	2·07	1·84	1·66	1·51	1·38	1·18	1·03	·92	·83	·69
1	·7854	2·670	9·44	6·28	4·71	3·77	3·14	2·69	2·36	2·09	1·88	1·71	1·57	1·35	1·18	1·05	·94	·78
1$\frac{1}{8}$	·9940	3·380	11·90	7·95	5·96	4·77	3·98	3·41	2·98	2·65	2·39	2·17	1·99	1·70	1·49	1·33	1·19	·99
1$\frac{1}{4}$	1·2272	4·172	14·7	9·8	7·36	5·89	4·91	4·21	3·68	3·27	2·95	2·68	2·45	2·10	1·84	1·64	1·47	1·23
1$\frac{3}{8}$	1·4849	5·049	17·8	11·9	8·91	7·12	5·94	5·09	4·45	3·96	3·56	3·24	2·97	2·55	2·23	1·98	1·78	1·48
1$\frac{1}{2}$	1·7671	6·008	21·2	14·1	10·60	8·48	7·07	6·06	5·30	4·71	4·24	3·86	3·53	3·03	2·65	2·36	2·12	1·77

CHAPTER XII

FOOTING SLABS

The footings included in this chapter are isolated footings to columns and footings between two columns. The question of continuous or " raft " foundations is in Chapter XXIX.

The design of footing slabs is complicated by the fact that they are very thick in proportion to their length and breadth and any application of the ordinary laws of flexure to such cases is somewhat uncertain. It is not surprising that there are several different methods of looking at the problem more particularly as regards the bending stresses produced. The author gives first of all a fairly conservative method, then a method sometimes used by competitive designers.

Punching Shear.—Let Fig. 97 represent a circular disc, of thickness

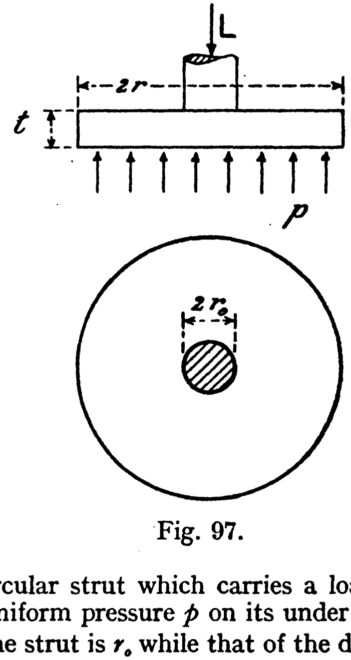


Fig. 97.

t, supporting a circular strut which carries a load L. The disc is supported by a uniform pressure p on its under surface.

The radius of the strut is r_0 while that of the disc is r.

We have $\pi r^2 p = L.$

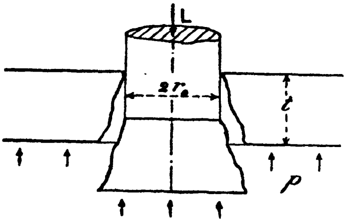


Fig. 98.

Now if t is small and L is large the strut may punch through the disc somewhat as shown in Fig. 98. The actual fracture will, of course, be irregular as drawn but it is found that the safe value of L can be expressed by assuming that the strut punches out a clean round hole of radius r_0 and that the intensity of shear is uniform over the surface of this hole.

If s_p is the safe value of this shear which is known as punching shear the maximum value of L is given by

$$L = s_p \times 2\,\pi r_0\,t + \pi r_0^2\,p$$

and

$$p = \frac{L}{\pi r^2}$$

$$\therefore \quad L\left(1 - \frac{r_0^2}{r^2}\right) = 2\,\pi r_0\,t\,s_p.$$

The condition of punching shear can only exist immediately underneath the column. If we draw lines at 45° as in Fig. 99 then outside these lines we must reckon the shear as beam shear at about 75 lbs. per sq. in. on the $b\,j\,d$ area as explained in Chapter IX.

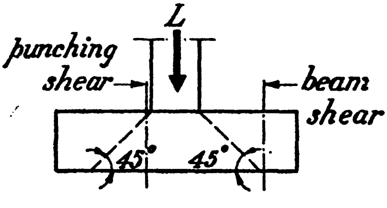


Fig. 99.

Bending Stresses Analysis.—Let Fig. 100 represent a square footing of side b carrying a square column of side c. The load on the column is L.

The intensity of pressure on the ground is assumed to be uniform and equal to p.

In order to satisfy the punching shear

$$L\left(1 - \frac{c^2}{b^2}\right) = 4\,c\,t\,s_p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(This assumes that the intensity of punching shear is uniform round the periphery of the column.)

Taking a section $x - x$ through the centre of the system. By symmetry there is no shear force on this section. The centre of the applied load on top is distant $\frac{c}{4}$ from the centre line (assuming uniform pressure on the column) while the centre of the applied load underneath is distant $\frac{b}{4}$ from the centre line. (Taking half the footing only.)

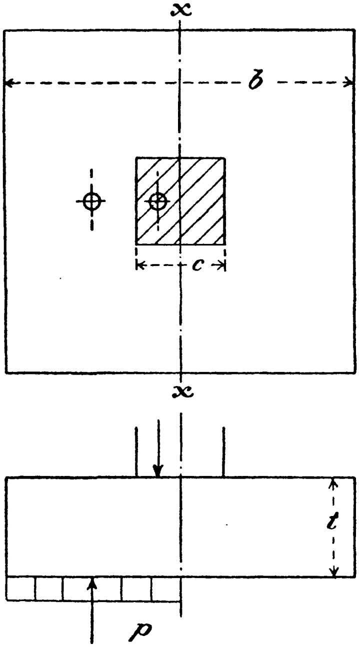


Fig. 100.

The total bending moment on the section is

$$\frac{L}{2} \times \frac{1}{2} \frac{(b-c)}{2}$$

$$= \frac{p \times b^2}{4} \cdot \frac{(b-c)}{2}.$$

The *average* bending moment is

$$\frac{p \times b^2}{4} \frac{(b-c)}{2 \cdot b}$$

$$= \frac{pb^2}{8}\left(1 - \frac{c}{b}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Up to this point our reasoning is quite sound but the following difficulties now arise :—

(1) The moment calculated is the average and not the maximum.

(2) The addition of a monolithic column will strengthen the central portion where the intensity of moment is greatest to an extent which can scarcely be calculated accurately.

(3) Solid concrete footings without any bars at all may be constructed provided the projection beyond the column face is less than half the thickness of the footing, as the shear-tensile strength of the concrete is sufficient to carry all the stresses. If we follow the usual reinforced concrete theory all such footings would fail as we assume no tensile strength in the concrete. The object of this discussion is to find how many bars are required in a practical footing. Academic considerations are of minor importance.

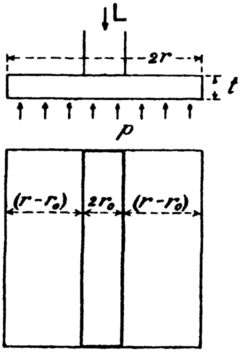


Fig. 101.

In equation (2) if c is small compared with b then

$$\frac{pb^2}{8}\left(1-\frac{c}{b}\right)=\frac{pb^2}{8}\text{ very nearly} \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

Taking the conditions stated above and striking a rough average it is usual to assume that the *maximum* bending moment on the slab is $\frac{pb^2}{8}$. In the case of a wall footing where the wall extends right across the slab as in Figure 101 we may safely assume that the maximum moment occurs at the face of the wall and is equal to

$$\frac{p \times (r - r_0)^2}{2} \text{ per unit length of wall.}$$

Application. Small Footings.—The reader will find that the thickness of the footing is most often settled by the punching shear, the necessary effective depth for bending strength being less than the thickness to resist punching. Small square footing slabs are often required. Tables of their sizes, etc., may be kept ready to hand.

This introduces a difficulty as the size of the column must be known before we can design for punching shear. The sizes shown in Table F I have been calculated as follows.

A bearing pressure and size of footing have been assumed. This gives us the maximum load such a footing will take. The column has been taken as carrying this maximum load and assumed as stressed uniformly to 600 lbs. per sq. in. The footings have been designed for stresses of 750 and 18,000 with $m = 18$ and 150 lbs. per sq. in. punching shear. In choosing the bars sufficient grip length must be allowed so that, with a bond stress of 100 lbs. per sq. in., the diameter must not exceed one-ninetieth of the width of the footing slab, for a stress of 18,000 lbs. per sq. in. For very small footings this may mean a large number of very small bars and it is better to use bigger bars working to smaller stresses. Some designers space the bars closer together under the column, but this is unnecessary in small footings.

Suppose we have a square footing as in Fig. 100, 6 ft. 0 ins. × 6 ft. 0 ins. in plan working to 3 tons per sq. ft. on the ground. The maximum load it can support is 108 tons or 242,000 lbs.

Assuming the column is stressed to 600 lbs. per sq. in. and has 1% of steel it will carry (Chapter VI) :—

$$c^2 \times 600 + \frac{c^2}{100} \times 12{,}900 = 729\,c^2 \text{ lbs.}$$

Size of column say $\sqrt{\dfrac{242{,}000}{729}}$ say 18″ × 18″

For punching shear.

$$242{,}000\left(1 - \frac{18^2}{72^2}\right) = 4 \times 18 \times t \times 150 \quad . \quad . \quad . \quad (1)$$

Min$^{\text{m}}$. $t = 21$ ins.

The average moment per 12 ins. width

$$= \frac{6720 \times 6^2}{8} \times 12 \text{ lbs.-ins.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

$$= 363{,}000 \text{ lbs.-ins.}$$

Min$^{\text{m}}$. d for bending at $138\,bd^2$

$$= \sqrt{\frac{363{,}000}{138 \times 12}} = 14{\cdot}8 \text{ ins.}$$

As t must be not less than 21 ins., d will be about 18¾ ins.
lever arm will be about 0·88 × 18¾ ins = 16·5 ins.

$$A_s \text{ per foot width} = \frac{363{,}000}{18{,}000 \times 16{\cdot}5} = 1{\cdot}22 \text{ sq. ins.}$$

Total A_s for a width of 6 ft. 0 ins. = 7·32 sq. ins.

To give 45 diams. grip-length the diameter should not exceed $\dfrac{72''}{90} = 0{\cdot}8$ in. say ¾-in. bars.

7·32 sq. ins. = say 17 bars ¾-in. both ways.

The footing is shown in Fig. 102.

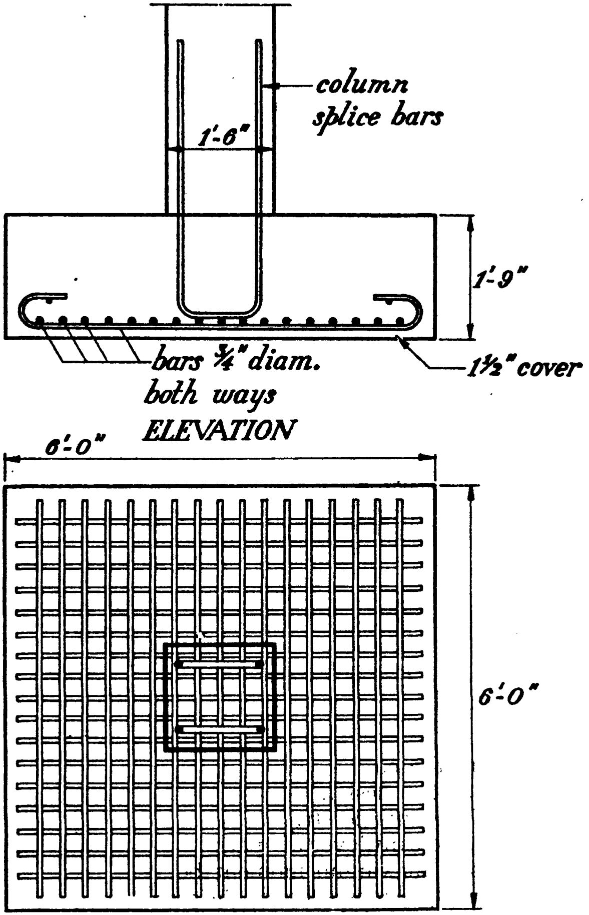


Fig. 102.

Quantities in Small Footings.—As the shear and bending moment diminish towards the edge of the footing slab the concrete may be sloped off as shown by the dotted line on the diagram at the head of Table F I. It is obvious that this method saves concrete but it also involves extra labour as the slopes will require centering on top if steeper than about 1 in 4 (slopes of 1 in 3 can be concreted without top boards if very dry concrete is used but this also entails extra trouble and expense). If the holes excavated for the footings are wet or awkward to get at, a flat top is the cheapest form. If the designer is keen on showing a saving in quantities (as distinct from a saving in total cost) he will use a sloping top. In good dry ground the holes may be dug exact size and no centering is required round the edges.

It is most convenient to place small vertical starting bars in the footing which project about 25—35 diameters. The footing can then be concreted without trouble and the column boxes stood on the concrete when this is set (this is another point in favour of using a flat top). If the main bars for the first length of column are taken right down into the footing they must be placed before the footing is concreted which means that some method of supporting them must be devised before the column boxes are set up. This is a nuisance and, from the contractor's point of view, short starting bars are an economy in cost although they call for a little extra steel.

Alternative Method of Design.—This method which is sometimes employed in competitive design is illustrated in Fig. 103.

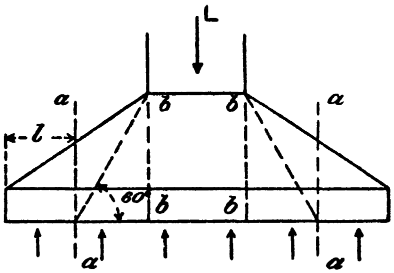


Fig. 103

The load *L* from the column is assumed to "spread" at 60° as shown. (Based on the design of solid concrete footings without reinforcement.) The outside strip of footing for a width *l* is then calculated as a simple cantilever of span *l* it being assumed that the maximum shear and maximum moment both occur at the section *a* — *a* (?) Some designers assert that the punching shear should be reckoned on the area *a* — *a*. This is based on a misconception of the safe punching shear. It is perfectly true, as already mentioned, that a footing slab tested to destruction does not punch through on a smooth parallel surface yet in such a test the result is stated by dividing the total safe load by the area of the sides of the hole *b*—*b* as shown in Fig. 103. Therefore in designing a footing, where the process is reversed, a similar area should be taken. The shear on areas *a* — *a* is not punching shear but ordinary shear and must not exceed about 75 lbs. per sq. in.

The depth *t* should be calculated as for the first method. (Fig. 100.)

In Figure 103 it will be seen that the moment at sections (*a* — *a*) is small and it appears that the maximum moment really occurs under the centre.

The distance $(a - a) = c + \frac{2}{\sqrt{3}}\, t.$

The footing may be looked on as a series of beams having a uniform load on the underside extending right across and a uniform load on top extending a distance $(a - a)$.

The maximum moment which occurs at the centre is

$$p \times \frac{b}{2}\left(\frac{b}{4} - \frac{(a-a)}{4}\right) \text{ where } (a - a) \text{ is in feet.}$$

It will readily be seen that this gives less steel than the first method—less than half as a matter of fact.

The reader will find that the greater the load on the ground the smaller the steel required as calculated by this method owing to the increase in the distance $a - a$ relative to b. It seems in this method that the corner parts of the slab are not very well covered and that the method would only apply strictly to a wall footing. Arguing from the similar case of the column heads (inverted) in a flat-slab floor it seems that the "spread" assumed is justified and many practical designers put less steel in their footings than shown in Table F I, particularly in thick footings.

(Table F I is based on the first method.)

If we take the punching shear at 150 lbs. per sq. in. as in the first method and assume a 60° spread calculating the bending moment at the centre we shall have the same thicknesses of concrete as shown in Table F I but the steel areas will be reduced as shown :—

$$\text{For } p = 2 \text{ tons}\begin{cases}\text{First method} & A_s = 0{\cdot}166\, b \text{ sq. ins. per foot}\\ \text{“ Spread ” } 60^\circ & A_s = 0{\cdot}086\, b \text{ sq. ins. per foot}\end{cases}$$

$$p = 3 \text{ tons}\begin{cases}\text{First method} & A_s = 0{\cdot}206\, b\\ \text{“ Spread ” } 60^\circ & A_s = 0{\cdot}085\, b\end{cases}$$

$$p = 4 \text{ tons}\begin{cases}\text{First method} & A_s = 0{\cdot}244\, b\\ \text{“ Spread ” } 60^\circ & A_s = 0{\cdot}081\, b\end{cases}$$

where b = side of footing in feet.

There are other methods used by various designers which give intermediate results.

Table F I.—This table is based on the first method put forward. Whatever method of design the reader adopts he may draw up a similar table as such small footing slabs are frequently required.

It is interesting to compare the moments given in this discussion with the stresses in a thin circular elastic plate (see Morley's *Strength of Materials*). The reader will find the calculated maximum stresses much in excess of those employed here.

Stepped Footings.—Occasionally (although seldom nowadays) a design similar to Fig. 104 is put forward where the footing, instead of being sloped off, is stepped off.

It is clear that the total moment on section $x - x$ is the same as the moment in Fig. 100, that is $\frac{pb^2}{8}\left(1 - \frac{c}{b}\right)$ but all this moment must be

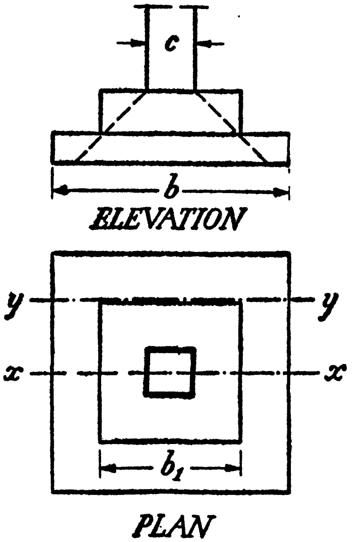


Fig. 104

taken by a rectangular beam section whose width is b_1. The cross section on the vertical plane $y - y$ is designed against the moment on a cantilever of span $\frac{b - b_1}{2}$.

As previously pointed out all sections outside the 45° lines must be designed for *beam* shear at about 75 lbs. per sq. in. on the $b\,j\,d$ area. Possibly on a contract where there are very many large footings all of similar size it might pay to step off or slope off the top (provided always that the shear stresses and bending stresses are covered) but in all other cases the practical difficulties will cost more to overcome than the cost of material saved.

It may be said in conclusion that the quantities in column footings do not as a rule form a very large proportion of the total quantities. If the quantities in a structure must be reduced it usually pays to devote one's energies to the larger items of the superstructure where alternative spacing of beams, etc., may result in considerable saving. Moreover, it must be remembered that failure of a column footing would involve a large part of the structure and be exceedingly difficult of repair, whereas failure of one member of the superstructure would be less important and may be more readily risked.

Footing Beams.—Where two columns or a row of columns stand close together a combined footing may be employed. Such a combined footing has the additional advantage of serving as a tie to the feet of the columns. This latter consideration in structures like water-towers, where the columns carry heavy bending moments, is important. The most usual form of footing-beam is an inverted T-section

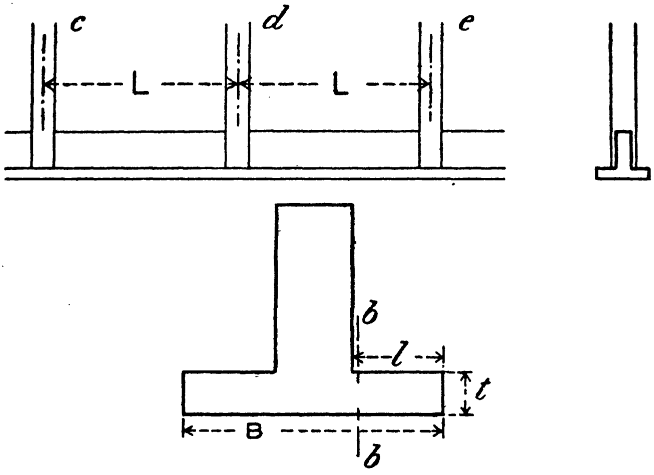


Fig. 105.

In Fig. 105 the upper part shows the elevation and section of such a row of columns while the lower part shows a larger scale section of the beam.

The overall breadth of the flange B is determined from the total load on the column, the spacing of columns and the safe bearing power of the foundation. Thus if the columns were 15 ft. 0 ins. apart and each carried 120 tons when fully loaded the ground being good for a load of 2 tons per sq. ft. we should have

$$2 \times B \times L = 120$$

$$30\,B = 120 \quad \text{i.e.,} \quad B = 4'\,0''.$$

The flange of the T-beam will act as a cantilever of span l, the maximum moment occurring at section b — b. Care must be taken in choosing the bars in the flange to make them sufficiently small to have sufficient grip length. Also in choosing a thickness it must be remembered that this flange has to resist the compression due to the bending moment in the beam. The method of designing such footing beams must be postponed until we have dealt with the design of rectangular and T-beams.

Distribution of Pressure on Footing Beams.—If the foundation were elastic and uniform throughout and the beam were extremely stiff then the intensity of pressure would vary uniformly along the beam. In the particular case when the beam is centrally loaded, that is when the centre of gravity of all the loads on the beam from above lies directly over the centre of gravity of the foundation area, the intensity of pressure would be uniform all over.

If however we assume that the beam is elastic and yields under pressure then the intensity of pressure on the foundation would be greater directly under the columns than in mid-span. The theoretical solution of the case of a uniform elastic beam resting on a uniform elastic foundation is closely akin to the case of " restraint " in circular tanks which is discussed in Chapter XX. Unlike this latter problem, however, the distribution of pressure on a uniform elastic foundation, since such a foundation never occurs in practice, is devoid of all practical interest.

In fairly uniform ground it is probable that the intensity of pressure is greatest under the columns, whereas a patchy or mixed ground (say for example clay with large boulders) may have a particularly hard patch under the centre of one span and comparatively soft patches under the columns. Another factor in determining the bending moments is the fact that certain columns in a row may be fully loaded while others are only partially loaded. The system of loading is *not* equivalent to the loading on a floor inverted, as is sometimes stated. Thus in Fig. 105 column *d* might be fully loaded while columns *c* and *e* carried nothing but dead weight. If columns *c*, *d* and *e* carried an inverted floor it would be necessary to load the whole of spans *c d* and *d e* in order fully to load column *d*. This would necessitate loading columns *c* and *e* also, to at least half their full load. Again, if *c* and *e* are unloaded then these two columns cannot carry any load whatever from spans *c d* and *d e* because they have no load to resist any upward thrust from the foundation. If the row of columns consisted of three columns only, such as *c*, *d* and *e*, and *d* the centre one was loaded while *c* and *e* were not loaded, then the footing would act as a double cantilever about the foot of column *d* because columns *c* and *e*, having no load capable of resisting an upward pressure, would be powerless to act as points of support to the beam. This state of affairs appears somewhat alarming at first sight but is really not very serious as in such a case the foundation pressure would concentrate near the foot of the centre column and reduce the cantilever bending moment. It serves to illustrate the fact that the distribution of pressure on combined footings is a difficult and uncertain subject to solve. It is abundantly clear, on the other hand, that we must fix on a method for designing such cases which will make a fair allowance for all the factors involved.

If the maximum pressure on the ground under full loading of all columns be worked out and the beams be taken as continuous beams, supported by the feet of the columns and carrying a superload equal to the maximum pressure, a satisfactory design will result.

For example, take the case we have already quoted in connection with Fig. 105.

Columns *c*, *d*, *e*, etc., under full load carry each 120 tons while $L = 15' \, 0''$.

The load per foot run on the footing beam is $\frac{120}{15} = 8$ tons.

From Chapter X it appears that a bending moment of $\frac{wl^2}{12}$ should be provided for at mid-span and support for interior bays

$$\frac{wl^2}{12} = \frac{8 \times 15^2}{12} \text{ tons-feet}$$

$$= \frac{8 \times 15^2 \times 12}{12} \times 2240$$

$$= 4{,}040{,}000 \text{ lbs.-ins.}$$

$$\text{Shear at columns} = \frac{wl}{2} = \frac{8 \times 15}{2} = 60 \text{ tons}$$

$$= 134{,}000 \text{ lbs.}$$

The bending moment in the end spans, if all the spans were of the same length, would be $\frac{wl^2}{10}$.

The reader will find that the shear is generally the factor which decides the size of footing beams.

In this case, allowing for the fact that the beam will be deep in proportion to its span we may take 250 lbs. per sq. in. on the *b j d* area as the limiting total shear.

The *b j d* area would have to be

$$\frac{60 \times 2240}{250} = 535 \text{ sq. ins.}$$

If we make the depth of the beam equal to about three times its width (see Chapter XIV) a beam 14 ins. wide by 40 ins. overall would suffice.

As will be seen from the chapter on T-beams this is a much larger section than would be required to resist the bending moment alone.

Eccentrically Loaded Footing Slabs.—If the bending moment causing the eccentricity may act in any direction, such as a bending moment due to wind pressure for example, a square footing may be used. For such cases a size can be chosen from Table F I or some similar table by working out the maximum pressure on the ground and taking the size which is given in the table for an average pressure of similar magnitude. For example suppose our footing slab is 8 ft. 0 ins. × 8 ft. 0 ins. in plan and that it carries a direct load of 96 tons and a bending moment of 43 tons-feet.

Maximum intensity of pressure

$$= \frac{96}{8 \times 8} \text{ direct} = 1{\cdot}5 \text{ tons per sq. ft.}$$

$$+ 43 \times \frac{6}{8 \times 8^2} \text{ due to bending} = {\cdot}5 \text{ ,, ,, ,, ,,}$$

$$\overline{2{\cdot}0 \text{ tons per sq. ft.}}$$

For this case we should choose a size of 8 ft. 0 ins. × 8 ft. 0 ins. from Table F I to carry an *average* load of two tons per square foot, namely 23 ins. thick reinforced with 14 bars 1 in. both ways.

Should the bending moment which causes the eccentricity act in one direction only the footing slab may be increased in length in that direction.

In Fig. 106 such a case is shown

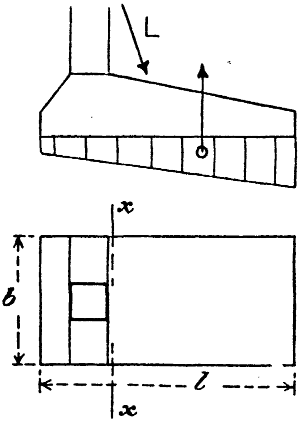
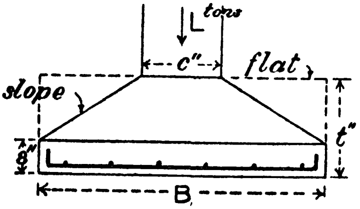


Fig. 106.

In such a long narrow footing the bending moment at any cross-section such as $x - x$ cannot vary much from one part of the section to another. By choosing $x - x$ to pass through the centre of the foot of the column and treating the part to the right of $x - x$ as a cantilever we may calculate the maximum bending-moment with a fair degree of accuracy. This bending moment will give us the long bars in the footing. The short bars which run across the footing may be put in at the same diameter and spacing as the long bars in that part of the footing directly under the column but should be spaced out towards the end of the footing. This cross steel is difficult to calculate and is best drawn in to eye as very little is required except immediately under the column. The punching shear in such a case may be taken on two faces only of the column (instead of on all four as in a square footing). Outside a line at 45° from the foot of the column (see Fig. 99) a condition of beam shear must be allowed for. The fact that the footing is eccentrically loaded implies a bending moment on the column. The designer must therefore assure himself that the column bars are well anchored in the footing.

TABLE F I.



$$M = \frac{pB^2}{8} \qquad C = \sqrt{\frac{L \times 2240}{729}}$$

$f_s = 18{,}000$

punching shear

= 150 lbs. per sq. in.

SAFE BEARING 2 TONS PER SQUARE FOOT ON GROUND

SIZE OF FOOTING	TOTAL LOAD TONS	ASSUMED SIZE OF COLUMN	t	STEEL BARS BOTH WAYS	TOTAL STEEL LBS	CONCRETE CUB. YARDS	
						FLAT	SLOPE
3′ 0″ × 3′ 0″	18	10″ × 10″	9″	14 Bars ⅜″	38	0·25	0·24
3′ 6″ × 3′ 6″	24·5	10″ × 10″	10″	11 „ ½″	65	0·38	0·34
4′ 0″ × 4′ 0″	32	10″ × 10″	12″	14 „ ½″	90	0·59	0·49
4′ 6″ × 4′ 6″	41	11″ × 11″	13″	17 „ ½″	119	0·81	0·64
5′ 0″ × 5′ 0″	50	13″ × 13″	15″	14 „ ⅝″	169	1·16	0·85
5′ 6″ × 5′ 6″	60	14″ × 14″	16″	17 „ ⅝″	224	1·49	1·07
6′ 0″ × 6′ 0″	72	15″ × 15″	18″	14 „ ¾″	292	2·00	1·34
6′ 6″ × 6′ 6″	85	16″ × 16″	19″	16 „ ¾″	357	2·46	1·63
7′ 0″ × 7′ 0″	98	18″ × 18″	20″	14 „ ⅞″	468	3·05	2·02
7′ 6″ × 7′ 6″	112	19″ × 19″	22″	12 „ 1″	554	3·80	2·40
8′ 0″ × 8′ 0″	128	20″ × 20″	23″	14 „ 1″	684	4·55	2·83
8′ 6″ × 8′ 6″	144	21″ × 21″	25″	12 „ 1⅛″	823	5·59	3·72
9′ 0″ × 9′ 0″	162	23″ × 23″	26″	14 „ 1⅛″	1014	6·50	3·90
9′ 6″ × 9′ 6″	180	24″ × 24″	28″	15 „ 1⅛″	1130	7·78	4·55
10′ 0″ × 10′ 0″	200	25″ × 25″	29″	17 „ 1⅛″	1335	8·95	5·15

SAFE BEARING 3 TONS PER SQUARE FOOT ON GROUND

SIZE OF FOOTING	TOTAL LOAD TONS	ASSUMED SIZE OF COLUMN	*t*	STEEL BARS BOTH WAYS	TOTAL STEEL LBS	CONCRETE CUB. YARDS FLAT	CONCRETE CUB. YARDS SLOPE
3′ 0″ × 3′ 0″	27	10″ × 10″	10″	12 Bars ½″	52	0·28	0·25
3′ 6″ × 3′ 6″	37	11″ × 11″	12″	14 ,, ½″	74	0·45	0·39
4′ 0″ × 4′ 0″	48	12″ × 12″	14″	17 ,, ½″	108	0·69	0·53
4′ 6″ × 4′ 6″	61	14″ × 14″	16″	22 ,, ½″	153	1·00	0·72
5′ 0″ × 5′ 0″	75	15″ × 15″	18″	17 ,, ⅝″	205	1·39	0·95
5′ 6″ × 5′ 6″	91	17″ × 17″	19″	21 ,, ⅝″	274	1·76	1·21
6′ 0″ × 6′ 0″	108	18″ × 18″	21″	17 ,, ¾″	352	2·33	1·52
6′ 6″ × 6′ 6″	126	20″ × 20″	23″	20 ,, ¾″	445	3·00	1·90
7′ 0″ × 7′ 0″	147	21″ × 21″	24″	17 ,, ⅞″	563	3·62	2·27
7′ 6″ × 7′ 6″	167	23″ × 32″	26″	15 ,, 1″	694	4·52	2·76
8′ 0″ × 8′ 0″	192	24″ × 24″	28″	17 ,, 1″	830	5·52	3·31
8′ 6″ × 8′ 6″	217	26″ × 26″	30″	15 ,, 1⅛″	1023	6·67	3·91

SAFE BEARING 4 TONS PER SQUARE FOOT ON GROUND

3′ 0″ × 3′ 0″	36	11″ × 11″	12″	14 Bars ½″	58	0·33	0·28
3′ 6″ × 3′ 6″	49	12″ × 12″	14″	17 ,, ½″	86	0·53	0·41
4′ 0″ × 4′ 0″	64	14″ × 14″	16″	20 ,, ½″	126	0·79	0·57
4′ 6″ × 4′ 6″	81	16″ × 16″	18″	18 ,, ⅝″	181	1·12	0·77
5′ 0″ × 5′ 0″	100	18″ × 18″	20″	21 ,, ⅝″	251	1·54	1·04
5′ 6″ × 5′ 6″	121	19″ × 19″	22″	25 ,, ⅝″	326	2·05	1·34
6′ 0″ × 6′ 0″	144	21″ × 21″	24″	20 ,, ¾″	414	2·67	1·70
6′ 6″ × 6′ 6″	169	23″ × 23″	26″	24 ,, ¾″	532	3·38	2·09
7′ 0″ × 7′ 0″	196	25″ × 25″	28″	20 ,, ⅞″	664	4·23	2·59
7′ 6″ × 7′ 6″	225	26″ × 26″	30″	18 ,, 1″	847	5·20	3·06

NOTE.—The weights of steel do not include short starting bars or column bars of any description.

CHAPTER XIII

RECTANGULAR BEAMS

Slabs.—These constitute the great majority of rectangular section beams which occur in practice, as isolated beams such as lintels or braces are comparatively rare. It is true that T-beams when subjected to reversed moment behave as rectangular sections since the concrete flange cannot safely be relied on to resist tension. This aspect of T-beam design, however, must be postponed until Chapter XIV, as it is impossible to go into the question without considering the conditions throughout the whole length of the beam.

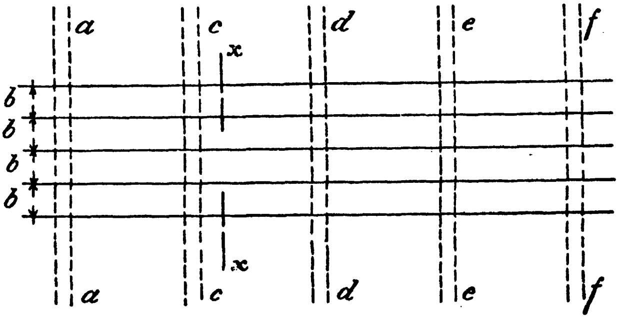


Fig. 107.

Only one case of bending will be treated here, namely, the case where the slabs consist of long comparative narrow panels and span in one direction only. Cases of square panels supported on all four sides or slabs supported at points (flat slab construction) are treated later. This case of simple spans is the most important and most generally occurring. The question has been further divided into floor slabs and wall slabs as the conditions of loading and construction differ slightly in the two cases.

Floor Slabs under a Superload.—Let Fig. 107 represent a series of long narrow floor panels in plan. These panels are supported on a series of beams or supports $a - a$, $c - c$, $d - d$, etc. Let us suppose that this floor has to support its own weight w_1 per sq. ft. and a super-load w_2 per sq. ft. Divide the floor into a series of equal strips by drawing parallel lines at right angles to the beams. If now the superload is arranged so that each one of these strips carries an equal amount of load, each strip will deflect in a manner similar to the adjacent strips. That is to say each strip may be treated as an independent beam as there will be no mutual inter-reaction between one strip and the next. Referring to Plates I and II we can see the distribution of superload that will cause the maximum bending moment at any point. It is clear that to produce the maximum possible bending moment at a section of the floor such as $x - x$ in the figure we must load every one of these small strips in such a manner as to cause the maximum bending moment in that strip at the section $x - x$. That is to say, each strip must be similarly loaded. It is perfectly fair, therefore, to treat each strip independently in such a case. (If the floor were called upon to carry an isolated point load, the particular strip on which the load actually occurred would be much more heavily loaded than its neighbours and would tend to deflect below the neighbouring strips, causing mutual reactions between one strip and the next. This case is treated later.)

In designing floor slabs the superloads are stated always as so much per square foot. The width b is chosen therefore as 12 ins. as this simplifies the calculations as the load per square foot on the floor becomes the load per foot run on a 12 in. strip.

The problem of designing such a floor resolves itself into the problem of designing a single rectangular section beam of width b.

Equal Spans.—As representing the easier and more common case this may be taken first. We have shown in Chapter X that the largest bending moments occur at points of support and at mid-span, the bending moments elsewhere being numerically less.

The maximum values of these moments are $+\frac{w\,l^2}{12}$ and $-\frac{w\,l^2}{12}$ for interior spans where $w = (w_1 + w_2)$.

Thus we see that it is necessary to know the value of w_1 (the self-weight of the floor) before we can calculate the bending moment. This value can generally be written down in any particular case by inspection. Having assumed a value for the self-weight of the floor and thence calculated the bending moments we are faced with the problem of choosing a section.

Referring to Chapter III we see that it is possible by varying the reinforcement to vary the strength in bending over a very large range.

Thus if we have a beam such that

b = breadth
d = effective depth
m = 18

and if we vary the steel in such a beam from zero to 3% top and bottom (which we may look upon as about the practical limit for a rectangular beam) we can vary the strength in bending (see Table RB VII) from zero to 400 bd^2 without exceeding stresses of 750 and 18,000.

Conversely if we know the value of M

$$\text{writing } R = \frac{M}{b\,d^2} \text{ (as in the tables).}$$

R may be varied from zero to 400 by varying the reinforcement.

$$bd^2 = \frac{M}{R}$$

$$d = \sqrt{\frac{M}{b\,R}}.$$

If we know M and $b = 12$ ins. we see that it is possible to obtain a whole range of values for d by varying R. We shall therefore have to look elsewhere for a deciding factor.

Firstly an overall depth of 4 ins. may be taken as a minimum for standard work, depths of 3½ ins. for floors and 3 ins. for roof slabs being sometimes employed in competitive design. The employment of top steel at mid-span sections may be ruled out on practical grounds. Referring to Table RB VII we see that the employment of large quantities of tensile steel alone are not very effective in resisting bending moment as the value of R increases very slowly after passing the value 138 bd^2.

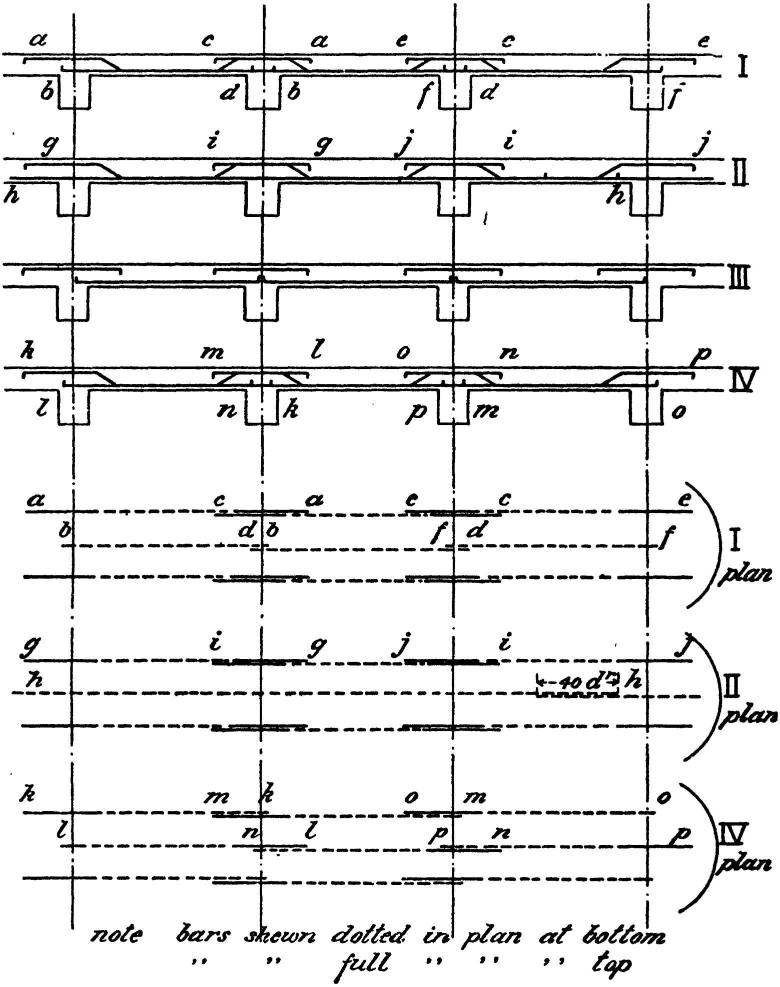
Considering all the circumstances we may assume as already mentioned in Chapters II and III that the value of $R = 138\ bd^2$ for stresses of 750 and 18,000 is the one to be aimed at ($m = 18$).

Thus we may write :—

$$d = \sqrt{\frac{M}{12 \times 138}}.$$

Arrangement of Bars. Equal Spans.—We now come to the most important point. How many bars shall we use, how long ought they to be and where shall we stop them. Looking at the subject from the purely theoretical standpoint, a large number of small bars is desirable. This would give an even distribution of stress with plenty of grip length and small adhesion stresses.

For a similar reason the bars should be as long as possible. The arrangement must also be contrived so as to provide for the maximum bending moments at all parts of the slab with the minimum total weight of steel. Looked at from the point of view of construction a small number of fairly large bars is much easier to place than a large number of small bars. The bars must be in manageable lengths and above all things must be placed in some simple and straightforward arrangement that is easily followed and which gets in the way of other bars and of men moving over the falsework as little as possible. The system, too, should be so arranged that it will keep in place during concreting with a minimum amount of blocking-up or wiring together. Fig. 108 shows the profile of several arrangements which are in actual use. Arrangement I has separate bars for each bay of floor. Bars $a - a$, $c - c$ and $e - e$ are bent up at both ends and carried over the points of support to resist the reversed moment, while bars *b* — *b*, *d* — *d* and *f* — *f* are carried through and project a short distance into the neighbouring spans. The plan shows the usual method of arrangement, the bent-



note bars shewn dotted in plan at bottom
" " full " " " top

Fig. 108.

up bars *a* — *a* and *c* — *c* being placed almost in the same straight line, the two bent-up ends being side by side.

This means that a section taken at a point of support would show first of all a pair of bars at the top of the slab, then a pair at the bottom and then a pair at the top and so on. In order to avoid this pairing up of bars, Arrangement I is sometimes laid so that a bent up bar $a - a$ in one bay lies in the same straight line as a straight bottom bar $d - d$ in the next bay.

Arrangement II is closely similar to Arrangement I the difference lying in the treatment of the straight bars. Instead of employing one length of bar for each span of floor the bottom bars are laid in long lengths and are lapped with one another about 40 diameters, the lap occurring at any point of a span. The bent up bars $g - g$, $i - i$ and $j - j$ are as in Arrangement I. Thus a section through Arrangement II at the support would show first a pair of top bars, then a single bottom bar, then a pair of top bars, etc. Since floor slabs are designed for $R = 138\ bd^2$, d being put in as the next practical size larger (than the calculated value for d) it follows that the addition of compression steel adds nothing to the strength of the slab as the percentage of tensile steel is 0·89% or slightly less (see Table RB VII).

Arrangement II is just as strong as Arrangement I and saves a little steel as the bottom bars have less frequent overlaps.

Arrangement III has no bent-up bars, the reversed moment being resisted by short lengths of top bar. The bottom bars may or may not be carried into the next span. This arrangement has the one advantage that it serves to use up short lengths of bar but otherwise is much inferior to Arrangements I, II or IV. It is capable of carrying a shear force of only about $75\ b\,j\,d$ lbs. against $100\ b\,j\,d$ lbs. carried by any of the other three arrangements. In addition to this the top bars are very difficult to support during concreting, whereas the other arrangements are to a great extent self-supporting. The author has known of cases where the top bars were pushed in after the slab had been concreted and had begun to set.

Arrangement IV consists of a series of similar bars all of which have one end bent up. They can be arranged so that two bent-up ends come together at the support or that one bent-up end lies directly over a straight end. The method of drawing a plan of the arrangement of bars in slabs is not generally necessary but may be adopted with advantage in complicated cases especially those having bars running in both directions or in cases where the designer is unable to visit the work as, for example, when work is designed in England for erection abroad.

In choosing between the four arrangements shown, number III should be avoided.

The author usually employs Arrangement II but laps the bottom bars always over the supports.

Standard Laps and Bends.—Hooks.—The question now arises as to the correct shape and length of the bars. First of all take the straight bars which pass at the support. At interior supports these are not needed as far as bending moment is concerned but are valuable for resisting shear.

The maximum stress they can possibly take is about 10,000 lbs. per sq. in. and a lap of 20 to 25 diameters past the point of maximum reversed moment is enough.

For ½-in. bars, each bar may be taken 1 ft. 0 ins. past the centre-line of the supporting beam, giving a total lap of 2 ft. 0 ins.

For equal spans the bent-up bars are carried past the support for a distance of one-quarter the span (sometimes only one-fifth the span in competitive work).

In Fig. 109 the distance x should, by theory, be not less than about $0{\cdot}1\ l$ and the distance y not more than about $0{\cdot}2\ l$ (see Tables CB V and CB VIII). For normal ratios of depth to span the bars may be bent up at 30°. For thick slabs such as foundation slabs they may be bent at 45°.

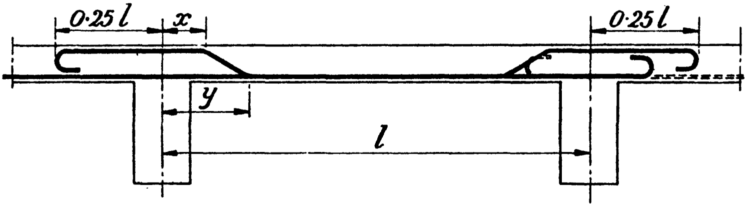


Fig. 109.

The length $0{\cdot}25\ l$ in Fig. 109 must, of course, be at least 45 diams. for a stress of 18,000 lbs. per sq. in.

In most slab work in existing structures the bent-up bars have small square hooks at the end. The straight bottom bars have also square hooks in about 50% of existing work, while in the remaining cases the bottom bars have no hooks at all. With modern high-shrinking cement all main bars in slabs should have efficient end-anchors such as shown in Fig. 109.

Equal Spans.—End Bays.—It has been shown in Chapter X that the moments in the end span and at the support next to the end support are about 20% higher than the moments for interior spans and supports. The methods of dealing with these increases may be divided into three cases.

I. Supplying increased steel and keeping the same thickness of slab.

II. Supplying a slightly thicker slab (adding say ½ in.) with the same or slightly increased steel.

III. Shortening the end-spans. (This rather begs the question although the reader will see what is meant.)

Methods I and II are really both violations of the assumption that $E\ I$ = constant for the whole system, which means that the increase in bending moment which should be allowed is somewhat in excess of 20%, as the effect of stiffening up one span is to increase the bending moments on that span.

This point is of little importance practically and a value of $\frac{w\,l^2}{10}$ may be adhered to.

It may happen, in choosing the depth of the slab for the interior spans, that a fair margin exists between the minimum calculated value and the next larger practical size.

Suppose the moment in the interior panels is

$$\frac{w\,l^2}{12} = 38{,}200 \text{ lbs.-ins.}$$

Working to 750 and 18,000 with $m = 18$

$$\text{min}^{\text{m}}.\ d = \sqrt{\frac{38{,}200}{138 \times 12}} = 4{\cdot}8 \text{ ins.}$$

With $\frac{1}{2}$-in. bars and $\frac{1}{2}$-in. cover the next larger practical size is a 6 in. slab with an effective depth of 5·25 ins.

Thus although we have aimed at a value of $R = 138$, we have actually a value of

$$R = \frac{38{,}200}{12 \times 5{\cdot}25^2} = 115.$$

Taking the moment in the end span

$$\frac{w\,l^2}{10} = 45{,}800 \text{ sq. ins.}$$

and keeping the same overall depth

$$R = \frac{45{,}800}{12 \times 5{\cdot}25^2} = 139.$$

Therefore if we use about 0·9% of tensile steel (see Table RB VII) we are quite safe.

On the other hand if the practical value of d for the interior spans corresponds exactly to 138 bd^2 then the value of R for the end span would be

$$138 \times 1{\cdot}20 = 166.$$

In order to provide this value without top steel we see from Table RB VII with $A_s' = 0$ that 1·75% of tensile steel would be required (or twice as much as is required for the interior spans).

Coming next to the question of the support next to the end support we see that by adopting Arrangement I in Fig. 108 (or using Arrangement II and lapping the bottom bars over this support) that the steel will here be the average of the interior and end spans and will occur at top and bottom, i.e., $A_s' = A_s$.

For example, in the case last quoted if the percentage of steel in the interior spans is 0·89% and in the end span 1·75% then the support next to the end support would have 1·32% of steel top and bottom. This would give a value of R about 215 bd^2 (Table RB VII) which is ample.

Dealing now with the next method of meeting the increased moments, i.e., by thickening the end spans. It is clear that we can thus easily cover the moments near mid-span but difficulty arises at the support. Fig. 110, where the difference in thickness has been exaggerated, illustrates this point. It will be seen that the bottom bars cannot be taken straight through from span to span and the additional thickness is not effective to resist the moment at the support, for section $x - x$ in the figure is only equal in thickness to the interior spans. This section must therefore be checked for $\frac{w l^2}{10}$.

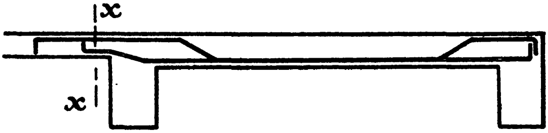


Fig. 110.

The slight additional weight per square foot due to thickening the end span need not be taken into account as usually such extra does not exceed ½ in., i.e., about 6 lbs. per sq. ft.

The third method raises the question, which we have hitherto avoided, of determining the length of the end span. The centre spans must be taken from centre to centre of beams unless the beams are unusually wide and held against rotation, but such a value is not applicable to end spans. For instance in Fig. 111 are shown two cases. In the upper one the end beam is narrow while in the lower one it is wide. If both these cases had the same clear span then the lower case would have a longer span centre to centre of beams. In the author's opinion it is fair to measure the end span from the centre of the last beam but one to the face of the last beam. In Fig. 111 this distance is marked " span ". (Most authorities say that the correct distance to take is " span " + 0·5 t.) The exact length of the span will depend eventually on the division of the whole site. If the overall length of a building is divided into even spaces it is clear that the end spans of the beams measured centre to centre of columns will be less than the interior spans by half the width of the external column. If such a method of spacing the columns is adopted it will be found easy to make the end spans of

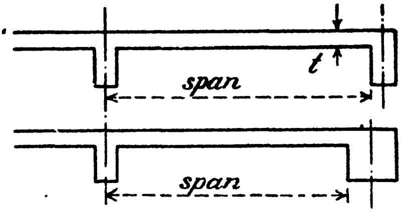


Fig. 111.

both slabs and beams somewhat smaller than the interior spans, thus reducing the value of the moment $\frac{w\,l^2}{10}$ for the end spans.

Of the three methods of treating end spans the first and last or a combination of the two should be adopted, as the second method means extra trouble in bending the bars and means using a special box for the last beam but one.

Unequal Spans.—As foreshadowed in Chapter X there is no easy method of arriving at the moments in cases of unequal spans. Even if the moments for an ideal elastic beam could be calculated easily for all cases, the application of these calculated moments to an actual beam and the deduction of a rational arrangement of bars would require thought and judgment. If the case of inequality is caused by the introduction of a single unequal span into a series of equal spans such as shown on Table CB IX we can arrive at the moments by comparing the actual case with this curve. The two cases which are worked out namely $E\,I$ = constant and $E\,I$ = a constant × (span)^3 may be taken as representing the two limits. In a practical case $E\,I$ would vary, but seldom to such an extent as (constant × span)^3. Even if the depth of concrete were kept constant the steel would be varied, while if the depth were varied it is unlikely that it would be varied in constant ratio to the span as questions of headroom might crop up, not to mention additional weight.

If the variation in span is small, i.e., less than 15% we may take each span separately and work to $\frac{w\,l^2}{12}$ or $\frac{w\,l^2}{10}$ according as we are dealing with interior or end spans. For any one support we may take

$$w\left(\frac{l_1 + l_2}{2}\right)^2 \times \frac{1}{12} \text{ or } w\left(\frac{l_1 + l_2}{2}\right)^2 \times \frac{1}{10}$$

respectively where l_1 and l_2 are the two spans meeting at the support in question.

Other cases must be treated on their merits. If the case in point is very complicated and very important the designer may have to work it out by the Theorem of Three Moments or the amended theorem given in Chapter X. If the case concerns only a small and relatively unimportant part of a large structure a covering value of the moments can be guessed at, particularly if only a preliminary design is being made.

Although top steel is always a nuisance it generally cannot be avoided when a small span adjoins a much larger one.

The occurrence of unequal spans is not very frequent in floor slabs as uniform spans lead to rapid and cheap construction and every effort is made therefore to keep to a uniform spacing of beams.

The example which is solved at the end of this chapter may prove instructive.

Point Loads on Floor Slabs.—When designing road bridges the question of concentrated loads on the bridge floor has to be faced. As already mentioned in the discussion of superloads, a slab under a superload which is stated to be so much per square foot can be treated as a series of rectangular beams placed side by side. A concentrated load on the other hand obviously causes bending in both directions, the load itself being at the centre of a depressed area.

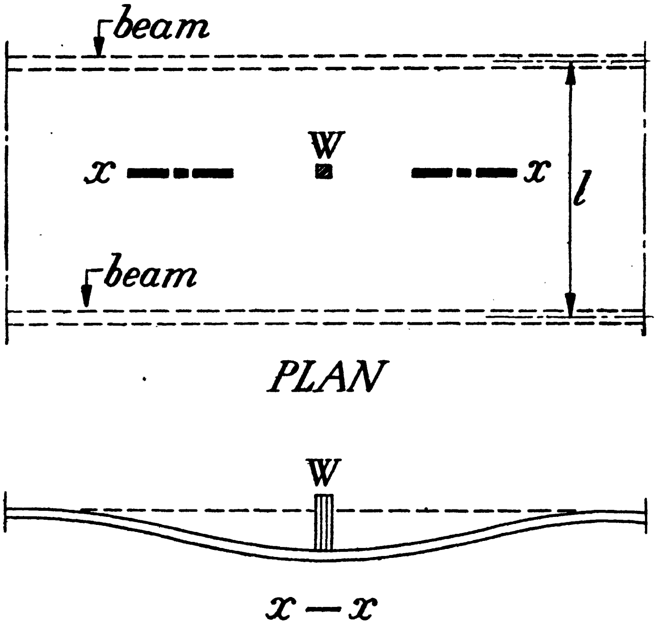


Fig. 112.

Suppose a panel of floor whose length is great in proportion to its span l carries a concentrated load W. A section $x - x$ taken through the load will show a maximum deflection under the load which decreases as we get further away from the centre of the load (see Fig. 112).

The mathematics of such a case is extremely difficult. A rough solution may be arrived at by a method similar to that used in Chapter XX. Combining this rough theory with results of tests and practical experience it is clear that the sideways curvature of the slab is proportional to the span.

If we have a concentrated load W in Fig. 113 covering an area $a \times a$ placed at mid-span in the centre of a long panel of floor slab of span l the maximum moment that it produces is the same as the moment caused by an equal load W spread out as a knife-edge load over a length $(a + 0{\cdot}6\,l)$ on a panel of floor slab whose total length is $(a + 0{\cdot}6\,l)$ as in Fig. 114. A point load on the mid-span of an interior span of a continuous beam having a number of equal spans causes a moment of about $\frac{Wl}{6}$. The moment per unit width of slab in Fig. 114 is $\frac{Wl}{6} \times \frac{1}{(a + 0{\cdot}6\,l)}$.

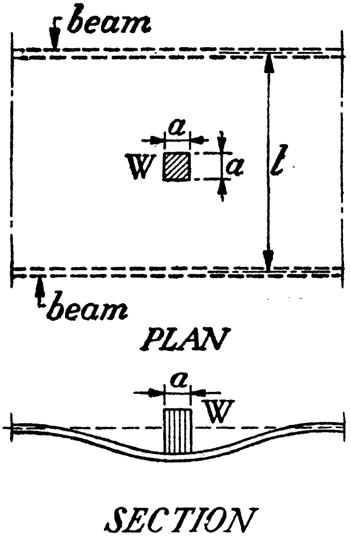


Fig. 113.

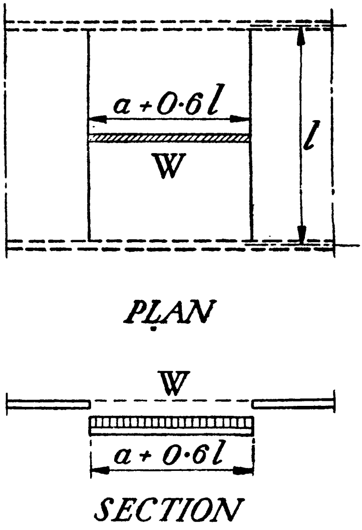


Fig. 114.

Suppose W is 10 tons applied on an area of 1 ft. 0 in. × 1 ft. 0 in. at mid-span of a long panel of floor slab spanning 9 ft. 0 ins., and suppose the floor is continuous over several similar panels on either side. The maximum moment produced by this load would be

$$\frac{10 \times 2240 \times 9}{6} \times \frac{1}{(1 + 0{\cdot}6 \times 9)} \text{ lbs.-ft.}$$

$$= \frac{10 \times 2240 \times 9}{6 \times 6{\cdot}4} \times 12 \text{ lbs.-ins.} = 63{,}200 \text{ lbs.-ins.}$$

It is clear from Figs. 112 and 113 that there is considerable transverse curvature of the slab and reinforcement is required parallel to the beams. This should be equal in amount to half the main reinforcement but need only be supplied over the middle half of the span. If, in the example above, the slab is 8 ins. thick and reinforced with ⅝-in. bars at 5 in. centres running from beam to beam, there should also be ⅝-in. bars at 10 in. centres running parallel to the beams but this need only be provided over the middle 4 ft. 6 in. of the slab (say six bars in each panel).

The foregoing treatment is intended for ordinary floor slabs spanning 8 ft. 0 in. or 10 ft. 0 in., where the concentrated loads are spaced 6 ft. 0 in. or 8 ft. 0 in. apart transversely. In plain slab bridges, consisting of a single slab spanning 30 ft. 0 in. or 40 ft. 0 in., carrying a series of concentrated loads about 8 ft. 0 in. apart, the transverse moment will only be a very small fraction of the main bending moment and nominal transverse reinforcement will suffice (say about 0·2%).

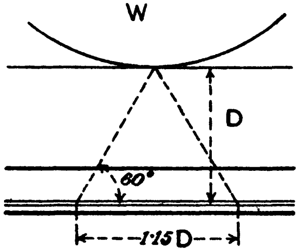


Fig. 115.

Spread due to Filling.—In Figures 112 and 113 the concentrated load is shown as covering a certain area. This is generally due to the provision of some sort of filling or road metalling over the reinforced concrete slab. For well-consolidated fill we may assume that the load spreads at 60° to the horizontal as in Fig. 115 and that this spread is continuous down to the tensile steel in the slab. For solid wheels the point of application at the road surface should be taken as a point to allow for inequalities in the surface. For pneumatic tyres a definite area of contact may be taken.

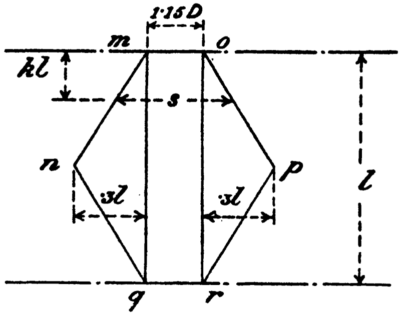


Fig. 116.

Road Bridge Floors.—A discussion of the proper loads to be provided for and the proper amount of impact to be taken is outside the scope of this book. It is clear from existing bridges that a slab 6 ins. thick is strong enough to prevent the heaviest modern wheel from punching through it. The worst moments are usually caused by loads at mid-span but other cases may be taken as in Fig. 116, which shows the plan of one span l.

Set off a distance equal to $1{\cdot}15\ D$ as in Fig. 115 (this may be increased for pneumatic tyres) at each line of support and draw divergent lines mn, op, qn, and rp which meet at mid-span such that $np = 1{\cdot}15\ D + 0{\cdot}6\ l$. Any concentrated load W which occurs at a distance kl from one support may be assumed to spread evenly as a knife-edge load along a straight line whose length is

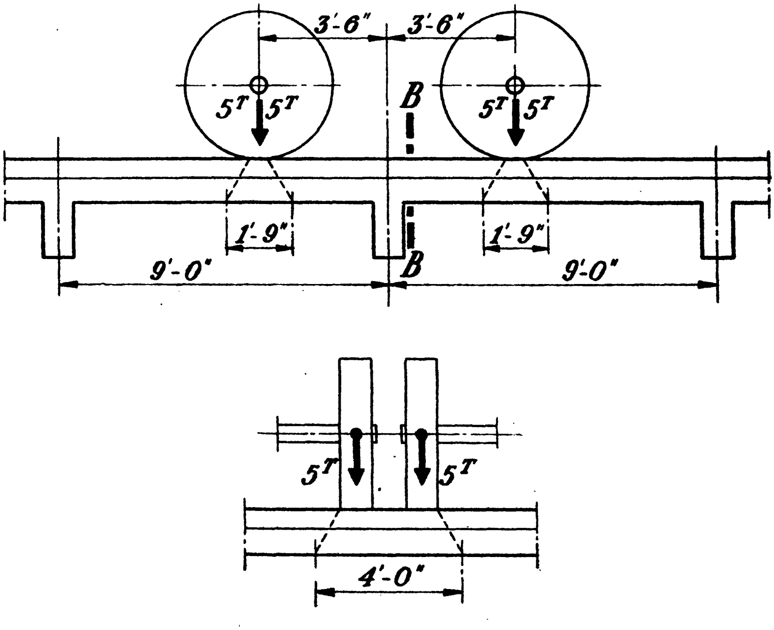
$$s = 1{\cdot}15\ D + 1{\cdot}2\ kl \text{ (up to } k = 0{\cdot}5)$$

Wheel loads on bridges often occur in pairs side by side, but enough has been said to indicate the general methods of allowing " spread " for such cases.

Having calculated the worst moment at mid-span the designer may check the worst moment at the supports but in general a standard arrangement of bars, as shown for superloads in Fig. 108, may be assumed to cover all cases without further calculation.

The proper amount of transverse steel is discussed above.

Example.—A bridge floor consists of a number of equal spans, 9 ft. 0 ins. centre to centre, each panel being long compared with its span. The slab is 8 ins. thick and is covered with 6 ins. of macadam.



B—B
Fig. 117.

It carries two pairs of wheels each weighing 5 tons placed as shown in Fig. 117. It may be assumed that the local spread due to the filling is 1 ft. 9 ins. × 4 ft. 0 ins. as shown. Find the moment per 12 ins. width of slab at section *B B* at the support due to these loads.

The two concentrated loads of 10 tons each are 3 ft. 6 ins. from the support which is about $0{\cdot}4\ l$. From Plate I the moment due to such a load is about $0{\cdot}085\ W\ l$.

The *total* moment due to both the 10-ton loads is

$$2 \times 10 \times 2240 \times 0{\cdot}085 \times 9 \text{ lbs.-ft.}$$
$$= 2 \times 10 \times 2240 \times 0{\cdot}085 \times 9 \times 12 \text{ lbs.-ins.}$$
$$= 412{,}000 \text{ lbs.-ins.}$$

There is a transverse spread of 4 ft. 0 ins. due to the width of the wheels and the spread through the filling.

Referring to Fig. 116 we can also count on a further spread of $1{\cdot}2\ k\ l$ due to transverse bending in the slab

$$1{\cdot}2\ k\ l = 1{\cdot}2 \times 3'\ 6'' = 4{\cdot}2 \text{ ft.}$$

Total sideways spread = 4′ 0″ + 4·2 ft. = 8·2 ft.

The moment per foot width is

$$\frac{412{,}000}{8{\cdot}2} = \mathit{50{,}100\ lbs.\text{-}ins.}$$

Slabs under Dead Load only.—It has been put forward that rectangular beams in general and slabs in particular conform to the general conditions of a perfect uniform elastic beam much more closely than T-beams and that therefore the bending moments which should be taken in designing them should follow the theoretical moments for such a case rather than the values of $\frac{wl^2}{12}$ at centre and support. (Compare Table CB V with Table CB VIII.) While the concrete is more uniform and less liable to shear distortion the conditions of support can never be relied on to follow the lines laid down in Chapter X. In the particular case where the slab carries a perfectly uniform load and is supported at constant intervals on a series of beams of similar span and similar design it might appear to a superficial observer that the supports, being equally loaded, would all sink a similar amount and thus all remain at the same level and further that, as the tangent to the slope remains unaltered at the points of support, these supports are for this particular case exactly equivalent to a series of frictionless point supports. While it must be admitted that the conditions of an actual structure under perfectly uniform loading are closer to the conditions of an ideal structure than are the conditions under partial load, it is idle to pretend that any actual structure of reinforced concrete is a uniform structure. The strength and elastic modulus of the concrete may vary three hundred per cent. from one part to another. Deflection of the shuttering, variations in placing the steel and arrangement of construction joints all are disturbing factors. Actual mistakes such as incorrect spacing of beams or the omission of a series of bars or stirrups need not be taken into account. These are accidents which cannot be covered by any theoretical argument or analysis and should therefore be ruled out of the discussion. We are concerned therefore only with those variations in construction which are inherent in all reinforced concrete work and which cannot be avoided.

It will be seen from the foregoing that an accurate determination of bending moments, since these depend on the relations between the deflections, is not to be looked for.

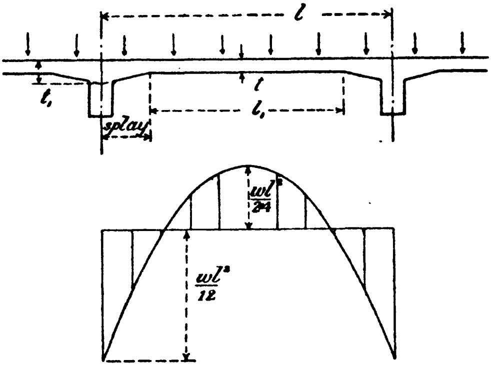


Fig. 118.

Splayed Sections.—The slabs in many existing structures where the loading is approximately uniform are constructed as shown in Fig. 118. The middle part of the span l_1 is designed for a moment of $\frac{wl^2}{24}$ while the slab is thickened out at the support. In general the thickness t_1 is about twice t so that the same steel run through will cover a moment of $\frac{wl^2}{12}$ at the support. The length marked " splay " in Fig. 118 is usually equal to $\frac{l}{6}$.

Splayed sections were most often used for retaining walls, bunker walls or silos. They are seldom seen in modern design except perhaps in heavy foundation work (see Chapter XXIX).

Wall Slabs.—The design of wall slabs in general is closely akin to the design of floor slabs. The main points of difference are :—

(a) The self weight of a vertical wall slab does not affect the bending moment.

(b) In a large number of cases the load may be applied on either face.

(c) The load is generally uniform in character and not liable to a hit-and-miss distribution as may happen with a floor superload.

(d) The load sometimes is applied on the face where the beam ribs occur. This introduces a tendency to tear the slab away from the beams.

(e) The provision of steel on both sides of a wall slab causes little additional difficulty. It is therefore possible to employ sections reinforced in both tension and compression more readily.

Wall slabs that are not provided with adequate splays should be designed for moments of $\frac{wl^2}{12}$ at mid-span and support for centre spans.

Where the slab serves as the outer wall of a building and is not liable to have merchandise piled against its inner face a 4 in. wall with 10 or 12 lbs. of steel per sq. yd. may be employed for small spans. For larger spans (where the panel exceeds 15 ft. 0 ins. floor to floor or 20 ft. 0 ins. from column to column) the wall should be made adequate to resist a pressure of 10 lbs. per square foot due to wind. For internal walls in similar circumstances a pressure of 5 lbs. per sq. foot should be taken.

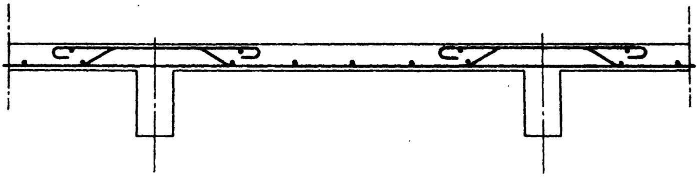


Fig. 119.

Transverse Reinforcement.—All slabs have some transverse steel (see Fig. 119). In the case of mesh panels (Chapter XVII) and floors carrying point loads, transverse steel is needed to help carry the load. Other slabs require transverse steel to help resist T-beam shear, to tie the main steel in position and to assist in controlling temperature stresses and shrinkage stresses. It is the last item which causes the most disturbing tension in modern work and the amount of transverse steel required depends on the total area of the floor, the height of the floor above the foundations and the type of cement used. With a small upper floor (not a roof) on flexible columns constructed of the type of cement in use about 1920, the transverse steel should be not less than 0·12% of the gross cross-section. With a similar floor built of modern cement the transverse steel should be not less than 0·2%. A large ground floor on very stiff columns built in modern high-shrinking cement should have not less than 0·3%. For a 5-in. floor slab the transverse steel required would be ⅜-in. every 18 ins., ⅜-in. every 11 ins., and ⅜-in. every 7½ ins. for the three types of floor just mentioned. If the main steel in a wall slab runs horizontally, little vertical steel is required as transverse reinforcement. A few fairly stiff bars are needed to support the horizontal steel during construction.

If the main steel runs vertically the transverse steel in bunker walls should be 0·12% to 0·2% and in water tank walls 0·2% to 0·3% depending on the length of wall and nature of the wall's foundations.

Quantities in Slabs.—The area of slab is taken off in square yards. The concrete for the slab is taken as extending over the whole floor or wall area, the projections only being taken as the beams. In a floor where the slab is 5 ins. thick and the beams 8 ins. × 20 ins. overall, the amount of concrete in each beam would be reckoned on an area of 8 ins. × 15 ins. section.

The amount of concrete per square yard of slab

in a 4 ins. slab is $\frac{4}{36}$ or 0·111 cub. yds.

in a 5 ins. slab is $\frac{5}{36}$ or 0·139 cub. yds.

in a 6 ins. slab is $\frac{6}{36}$ or 0·167 cub. yds.

and so on.

In a splayed section (see Fig. 118) the splays should be taken separately.

The centering is taken as if the slab extended over the whole area, the beam projections having been removed. Slabs which slope at more than 1 in 4 to the horizontal should have both sides included.

Thus we have :—

Floor slabs have 1·0 sq. yd. centering per sq. yd.

Wall slabs have 2·0 sq. yd. „ „ „ „

In the calculations of strength the steel is stated as so many square inches section per foot run of slab section at mid-span. In the quantities we want the weight of steel per square yard. We can convert the former into the latter by using a *steel factor.*

Take the arrangement of bars in Fig. 109. The bent-up bars run over a distance of 0·25 l into each adjoining span so that in plan the bar extends over a length of 1·50 l. In addition we must provide for two end hooks and allow a little extra for the two sloping pieces. Suppose l is 8 ft. 0 ins. and the bars are ½ in. diam. and the slab 5½ ins. thick. Round end hooks will each need about 4½ ins. and the sloping pieces will increase the length by about 1 in. each. The total length of bar would be

$$\begin{aligned} 1{\cdot}50 \times 8'\,0'' + 9'' + 2'' &= 12'\,11'' \\ &= 1{\cdot}62\,l \end{aligned}$$

Thus the bent-up bar is 1·62 times as long as the span of slab which it reinforces.

The bottom bar may run over three spans of 8 ft. 0 ins. and may be 26 ft. 0 ins. long with two end hooks of 4½ ins. each. The bar will therefore be 26 ft. 9 ins. long and will reinforce three bays of 8 ft. 0 ins. Its length is 3·35 l for three bays or 1·12 l per bay. Following Arrangement II in Fig. 108 these bars occur alternately so that the average length of bar required is

$$\frac{1{\cdot}62\,l + 1{\cdot}12\,l}{2} = 1{\cdot}37\,l$$

Now suppose we take a piece of floor slab 12 ins. wide and l long. The area of bars at mid-span is A_s sq. ins. as given in the calculations. The weight of a steel bar in pounds per foot is 3·4 times its area in sq. ins.

The average length of the bars is 1·37 l and their total weight in an area 1 ft. 0 ins. wide × l ft. long is

$$3{\cdot}4 \times A_s \times 1{\cdot}37\,l$$

The area in square yards of this piece of slab

$$\text{is } \frac{1'\,0'' \times l}{9}$$

The weight of steel per square yard is

$$\frac{3{\cdot}4 \times A_s \times 1{\cdot}37\,l}{\left(\dfrac{1'\,0'' \times l}{9}\right)} = 42\,A_s \text{ lbs.}$$

The value 42 is therefore the *steel factor* we require.

In most panels some bars will be left out where the main beams occur and a factor of 40 will suffice.

This figure applies to fully continuous panels. In a large floor the end panels will have less overlaps but may have heavier reinforcement. If we take 40 times A_s at mid-span in the interior panels this figure will cover the main steel in the whole floor. If the entire floor is only two panels wide a factor of 37 is enough and if only one panel wide a factor of 33 will do.

The distributors are laid in long lengths and lapped a distance of about 7% to 10% of their length. No distributors occur directly over the beams and a factor of about 32 will cover them. If a slab has ⅜-in. distributors at 12-in. centres, their area per foot of section is 0·11 sq. ins. (Tables B II). Their weight per square yard of floor slab will be about

$$0{\cdot}11 \times 32 \quad \text{say } 3\tfrac{1}{2} \text{ lbs.}$$

Using ⅜-in. distributors at 18-in. centres will require about 2 lbs. per sq. yd.

If Fig. 119 is one interior panel of a large floor and is reinforced with ½-in. bars at 5-in. centres and distributors ⅜-in. at 15-in. centres, the main steel has an area A_s of 0·47 sq. ins. and the distributors have an area of 0·088 sq. ins. per ft. run of section. (Table B II.) The total weight of steel will be about

$$40 \times 0{\cdot}47 + 32 \times 0{\cdot}088$$
$$= \text{say } 21 \text{ or } 22 \text{ lbs. per sq. yd.}$$

These factors, it must be understood, are only used in preliminary designs, estimates and tenders. When the job is detailed the exact weight of steel may be taken from the descriptive lists.

Slabs.—Conclusion.—Some works give tabulated values of strengths of floor slabs. The author has never found any advantage in using them and knows no one who does use them as it is easier to run out the values on a slide-rule than turn up a table.

Failure by excessive stress is practically unknown in floor slabs, probably owing to their power of spreading load transversely. (This statement does not refer to flat-slabs.)

There is one particular point which is worthy of mention as it is often " discovered " by writers on the subject.

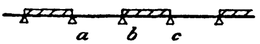


Fig. 120.

In the system of loading shown in Fig. 120 all the supports are evenly loaded and sink an equal amount and theoretically the span *ab* has a reversed moment of $\frac{w\,l^2}{24}$. The self-weight of the slab produces a bending moment $\frac{w\,l^2}{24}$. Therefore if the superload exceeds the self-weight the slab will actually suffer reversed moment. There is, of course, no danger of the floor actually bursting upward, but tension cracks in the span *ab* would throw more moment on span *bc* owing to the loss of continuity.

For heavy superloads, therefore, it is essential (so it is argued) to provide top steel right across every span. The reader by considering the conditions of loading and the relative sizes of spans and members in heavy warehouse design will see at once many objections to this argument. These need not be mentioned here as there is another consideration which permits of no argument. The vast majority of heavy warehouse floors hitherto built have been constructed without this " necessary " top steel and such construction has invariably proved satisfactory. To increase the area of steel in a form of design of proved economy is a direct violation of the very first principles of engineering. Owing to lack of practical data this omission of top-steel is not yet possible for unequal spans.

Bracing.—Apart from slabs the most frequent occurrence of rectangular beams is in braces.

Many structures are designed with an insufficient number of members properly to triangulate them, which means that the members in such designs have to carry bending stresses.

The cross-sections of such braces fall into two groups see Fig. 121.

(*a*) Those having one vertical row of bars.

(*b*) Those having two vertical rows of bars.

The number of rows of bars being counted at the normal section and not where laps occur.

The choice between these two sections will depend on the span and bending moment.

Ratio Breadth to Span. Breadth to Depth.

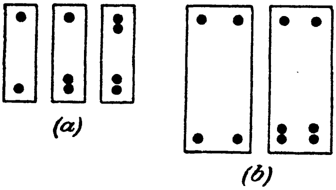


Fig. 121.

The ratio $\frac{\text{span}}{\text{breadth}}$ should not exceed 30 and the ratio $\frac{\text{depth}}{\text{breadth}}$ should not exceed about 3 although this ratio is somewhat more elastic. The ratio span : breadth is governed by considerations of buckling, either of the brace as a whole, should it be acting as a strut, or of the compression flange if the brace carries bending moment only. The ratio depth to breadth is limited by the difficulties of concreting a deep narrow beam and by the fact that increased depth shows no saving in steel beyond a certain point, for the sectional area of steel in a brace would never be less than ·7% top and bottom. The choice of a section may also be limited by the size of the members into which it frames at the ends. The absolute minimum breadth of section (*a*) is 4 ins. for bars up to ¾-in. diameter, about 5 ins. for 1 in. bars and 6 ins. for 1⅛ in. bars. This will be more apparent when the arrangement of bars is discussed. The absolute minimum for type (*b*) is a breadth of 7 ins. for ¾-in. bars, 8 ins. for 1 in. bars and about 10 ins. for 1⅛ in. bars.

It may be stated here that it is possible to *draw* sections much narrower than the breadths laid down above. For example, it is possible to draw a section 6 ins. wide, having two 1-in. bars side by side. Allowing 1 in. cover, such a section shows a clear space between the bars of 2 ins. This takes no account of the fact that, where the laps occur, four bars must be accommodated although it is still possible, by drawing the bars touching one another, to draw four bars side by side in a 6 in. breadth.

Steel bars are not easily pliable, especially those of large diameter and allowance must be made for " humouring " one pair of bars to pass inside the other pair at joints and laps, to say nothing of the difficulty of concreting. Emphasis is laid on the point as spacing and arrangement of bars is the most important point in design, and there is a great temptation for the designer when taking up the subject to place a large number of bars side by side in order to develop a greater bending strength by making the effective depth as great as possible (see the chapter on T-beams for a fuller discussion).

Before proceeding with the actual choice and arrangement of bars we shall examine one or two standard cases to discover what bending moments may be expected. A common application of bracing is shown in Fig. 122.

In such a case the position of the points of contraflexure may be assumed as occurring half-way along the length of each member. (This is not strictly correct but is near enough.) From Fig. 122 we may, assuming that the two legs of the trestle are equally stiff, write down the reactions at the points of contraflexure by halving the horizontal thrusts. This figure represents the lowest brace. If h_1 and h_2 are the half-heights of the middle and lower panels, we have :—

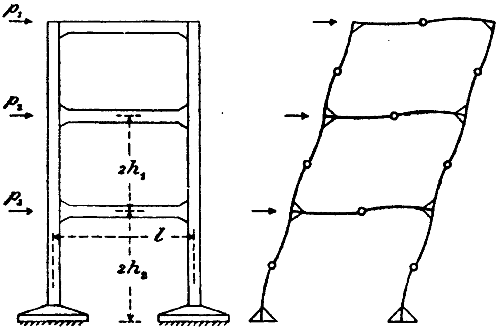


Fig. 122.

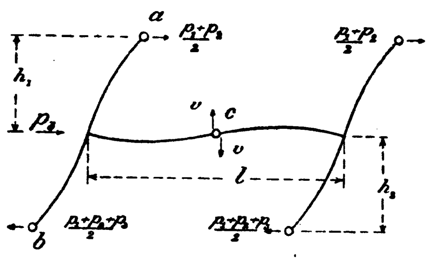


Fig. 123.

Bending moment in lowest tier of columns

$$= \frac{p_1 + p_2 + p_3}{2} \times h_2$$

Bending moment in middle tier of columns

$$= \frac{p_1 + p_2}{2} \times h_1$$

both these moments occurring at the junction of the brace and columns. By considering the equilibrium of the part *abc* we see that the bending moment in the brace at the junction with the column is equal to the sum of the moments in the columns. Taking moments about the junction point we have

$$\frac{p_1 + p_2}{2} \times h_1 + \frac{p_1 + p_2 + p_3}{2} \times h_2 = v + \frac{l}{2}$$

and $v \times \frac{l}{2} =$ the bending-moment in the brace.

All the members being loaded as cantilevers the bending moments vary uniformly from zero at the points of contraflexure to a maximum at the joints. In most cases such a trestle would carry a large superstructure the horizontal loads being due to wind. The braces would be spaced fairly evenly the bending moment on the lowest brace being about twice the bending moment on the lowest tier of columns.

The design of the joint presents the main difficulty in this case, as the bending moment in the column suffers a complete reversal in the depth of the joint.

In Figure 124 we see that the bars in the column may pass from compression to actual tension in the depth D while the bars in the brace which has to resist a moment M_3 where $M_3 = M_1 + M_2$ most certainly will change stress from the top of section $x - x$ (which, we will assume, is the critical section of the brace) to the bottom. The arrangement we have drawn is as good as can be managed and as will be seen the bars in the brace pass into the column and out again without laps. Thus the bars in tension at section $x - x$ and the bars in compression at section $x - x$ are one and the same bars and must suffer reversal of stress in the depth of the joint. With the arrangement as shown this is not a very serious

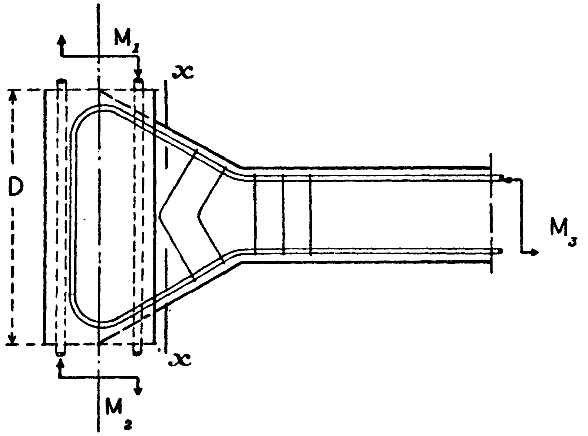


Fig. 124.

matter as the bars, owing to the bends are well anchored. The column bars however are straight and their grip length should be looked into. The intensities of stress on these bars will vary with each particular case. If the columns carry a large proportion of direct load and not much bending actual tension may not occur. In most cases the designer will find that a total change of stress from + 8,000 to − 8,000 equal to 16,000 lbs. per sq. in. will cover the worst case. This calls for a grip length of 40 diameters in the depth *D* in Fig. 124 and this value should be aimed at. The joint is a difficult one to construct but there is no question that the calculated bending moments are realized and must be provided for.

Two further arrangements of bars are shown in Fig. 125 the upper one showing the case where more than two columns occur in a line, the brace passing through. In this case the bending moment in the brace will be only one half the moment shown in Fig. 124. The lower one shows a stiffening brace put in to shorten the effective height of the column where no transverse loads occur.

Prior to the year 1910 most braces had splays at the ends at an angle of 45° to the member. From 1910 to 1920 angles of 30° were common as in Figs. 124 and 125 (upper figure). Most modern braces are made of uniform depth throughout as in Fig. 128 as this simplifies the steel and shuttering at the cost of a small extra volume of concrete. Only skew braces or very large braces would be given splays in modern design.

The secondary steel in braces since the shear is uniform and in most cases reversible in direction will consist of straight links. If the shear does not exceed 75 *b j d* the links may be spaced at a distance equal to the depth of the brace for a light brace. If the brace carries heavy shear or a heavy compression then a spacing of half the depth must not be exceeded. For a shear force only the

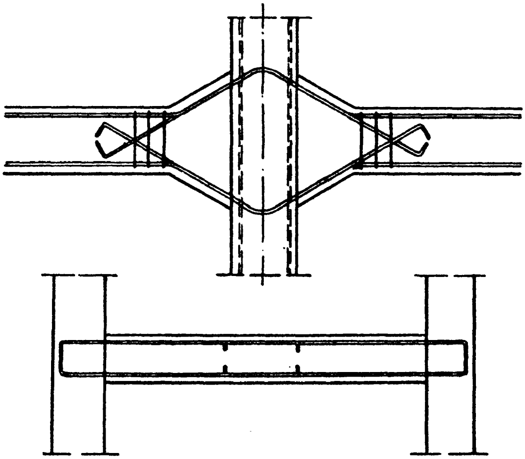


Fig. 125.

links may be calculated but for a strut a value of 0·20% by volume or upwards (see the chapter on sections under compression) should be provided.

Braces in such structures as pile jetties which are put in to resist accidental shocks are not amenable to design and are put in from experience, a section of 10 ins. × 14 ins. reinforced with four 1 in. bars being common for spans up to about 20 ft. 0 ins. For other braces a percentage of steel equal to about 1½% top and bottom, reckoned on the most highly stressed section will give a working size. From Table RB VII this makes $R = 250\,bd^2$. The bending moments due to the self-weight of the brace are, as a rule, small in comparison with the reaction moments from other members and may generally be ignored. The shear, however, since in general there are no bent-up bars requires attention as it must be all carried on the concrete and on vertical stirrups. Special stirrups must always be provided where sharp bends occur in tension bars. The depth of the brace in the centre of its span where its bending moment is small should not be less than one-third its maximum depth at the splay or less than about one-twentieth of the span.

Isolated Rectangular Beams.—Lintols to openings in walls may be designed for 138 bd^2 where b is equal to the thickness of the wall and d is so arranged that the overall depth is a whole multiple of 3 ins. If headroom is scarce the value of R may be increased and d correspondingly reduced by using steel top and bottom.

In crane girders the heavier loads usually call for steel top and bottom. Such a case is best solved by trying say four mid-span sections one for 138 bd^2, one for 200 bd^2, one for 300 bd^2 and one 138 bd^2 using a T-section with a narrow and deep flange. The approximate quantities for these four sections are readily written down and compared. In the special case of crane girders where some lateral load may be expected, steel must be provided at the top and the beam must be made wider to cover the sideways bending. Where no lateral load can occur we may try $b = \frac{d}{3}$.

$$\text{Since} \qquad M = R\,bd^2 = R\frac{d^2}{3}$$

$$\text{we have} \qquad d = \sqrt[3]{\frac{3M}{R}}.$$

For girders requiring lateral stiffness

$$\text{try } d = \sqrt[3]{\frac{2M}{R}} \text{ with } b = \frac{d}{2}.$$

The reader will now appreciate the value of being able to determine the bending strength of a rectangular section with steel top and bottom at a glance for then he can in a very short space of time design several varying sections. If none of his first attempts is exactly what he wants a second trial with some intermediate section should prove satisfactory.

In all rectangular sections reinforced with top steel the stirrups must be taken across the top of the beam to hold in the compression bars against sideways buckling and will resemble the links in columns. If this is not convenient separate ties may be used which can be put in after the main steel is placed. (In a T-beam reinforced for compression the top bars are prevented from buckling sideways by the slab.)

Arrangement of Bars in Rectangular Beams.—The problem of shear in rectangular beams is closely akin to the same problem in T-sections, although there is more steel to bend up in rectangular sections reinforced for compression owing to the fact that the tensile steel being stressed more lightly (see Tables TB I to TB VI for position of neutral axis) has a larger sectional area for similar depths and similar bending moments.

A similar arrangement of bars, etc., will suffice (see Chapter XIV) as indicated below.

Quantities in Rectangular Beams.—A rectangular beam having no top bars will have steel coefficients similar to those for a T-beam *q.v.* The quantities of concrete and centering are easily found after reference to Chapter XIV. Beams having top bars are best looked at as T-beams where the concrete flange is replaced by the top steel. The tensile steel may then be arranged in a manner similar to a T-beam according to the length of span, shear and relative numerical values of the bending moments at mid-span and supports. The weight of tensile steel can be found by using the steel coefficients in Chapter XIV the weight of compression steel being reckoned separately and added.

Holes in Slabs.—Large holes in slabs must be provided with trimmer beams but small holes such as manholes may be left untrimmed. The occurrence of a small hole will not greatly affect the deflections and curvatures and the normal reinforcement which would be used if no hole occurred should be followed. The bars which would normally cross the hole may be placed on either side. In addition extra bars as shown in Fig. 126 should be provided, the number varying with the extent and location of the hole and being decided on by eye in each case. In large floors of modern high-shrinking cement it is difficult to prevent shrinkage cracks starting from the corners of all openings.

Shear in Floor Slabs.—Only with heavy superloads (say over 5 cwts. per sq. ft.) is it necessary to check the shear stresses. With stresses of 750 and 18,000 with $m = 18$

$$\text{If } M = \frac{wl^2}{12} \text{ and } d = \sqrt{\frac{M}{138\,b}}$$

the shear stress on the $b\,j\,d$ area is less than 75 lbs. per sq. in. if w is less than 1450 lbs. per sq. ft.

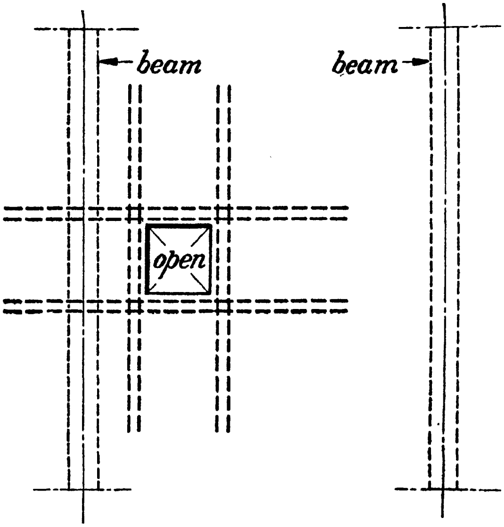


Fig. 126.

Examples on Floor Slabs.—The method of designing the sections is in Chapter II. The moments to be used in design are in Chapter X.

A floor slab consists of many spans of 7 ft. 0 ins. and carries a superload of 100 lbs. per sq. ft. Design an interior panel using stresses of 600 and 16,000 with $m = 15$.

$$\begin{array}{rcl} \text{Superload} & = & 100 \text{ lbs. per sq. ft.} \\ \text{Self-weight (say)} & = & \underline{48} \text{ ,, ,, ,, ,,} \\ & & 148 \end{array}$$

$$M = \frac{148 \times 7^2}{12} \times 12 = 7250 \text{ lbs.-ins.}$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{7250}{95 \times 12}} = 2{\cdot}52'' \text{ use 4 ins. slab with } d = 3{\cdot}25 \text{ ins. say}$$

$$A_s = \frac{7250}{16{,}000 \times 0{\cdot}88 \times 3{\cdot}25} = 0{\cdot}158 \text{ sq. ins. say } \tfrac{3}{8}\text{-in. every 8 ins.}$$

(= 0·17 sq. ins.)

Weight of steel per sq. yd

= 0·17 × 40 for main bars

+ 2 (say) for distributors

say 9 lbs. total.

Quantities per sq. yd.		
	Concrete = $\frac{4}{36}$	0·11 cu. yds.
	Steel	9 lbs.
	Centering	1·0 sq. yds.

A floor slab consists of many spans of 9 ft. 0 ins. and carries a superload of 2 cwts. per sq. ft. Design an interior panel, using stresses of 900 and 17,000 with $m = 14$.

$$\begin{aligned} \text{Superload} &= 224 \text{ lbs per sq. ft.} \\ \text{Self-weight (say)} &= \underline{54} \text{ ,, ,, ,, ,,} \\ &\quad \underline{278} \end{aligned}$$

$$M = \frac{278 \times 9^2}{12} \times 12 = 22{,}500 \text{ lbs.-ins.}$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{22{,}500}{174 \times 12}} \times 3{\cdot}28''$$

use $4\frac{1}{2}$ ins. slab with d = say 3·75 ins.

$$A_s = \frac{22{,}500}{17{,}000 \times 0{\cdot}858 \times 3{\cdot}75} = 0{\cdot}413 \text{ sq. ins.}$$

Say $\frac{1}{2}$-in. bars every $5\frac{1}{2}$ ins. (= 0·43 sq. ins.).

Concrete $\frac{4\frac{1}{2}}{36}$	0·125 cu. yds.
Steel 0·43 × 40 + 3 (say)	20 lbs.
centering	1·0 sq. yds.

A floor consists of two equal spans of 8 ft. 8 ins. and carries a superload of 300 lbs. per sq. ft. Design it for stresses of 750 and 18,000 with $m = 18$.

$$\begin{aligned} \text{Superload} &= 300 \text{ lbs. per sq. ft.} \\ \text{Self-weight (say)} &= \underline{66} \text{ ,, ,, ,, ,,} \\ &\quad \underline{366} \end{aligned}$$

$$M = \frac{366 \times 8{\cdot}67^2}{10} \times 12 = 33{,}100 \text{ lbs.-ins.}$$

$$\text{Min}^{\text{m}}.\quad d = \sqrt{\frac{33{,}100}{138 \times 12}} = 4{\cdot}46 \text{ ins.}$$

use $5\frac{1}{2}$-in. slab with d = 4·75 ins.

$$A_s = \frac{33{,}100}{18{,}000 \times 0{\cdot}857 \times 4{\cdot}75} = 0{\cdot}45 \text{ sq. ins.}$$

Say $\frac{1}{2}$-in. bent bars every 8 ins. = 0·29 sq.-ins.
$\frac{3}{8}$-in. straight bars every 8 ins. = 0·17 ,, ,,
0·46

Concrete $\frac{5\frac{1}{2}}{36}$	0·153 cu. yds.
Steel say 0·46 × 37 + 3	20 lbs.
Centering	1·0 sq. yds.

A foundation slab on clay soil consists of several spans of 10 ft. 0 ins. and carries an effective upward load of 10 cwts. per sq. ft. Design an interior panel for stresses of 750 and 18,000 with $m = 18$.

$$\text{Moment (see Chapter XII)} = \frac{wl^2}{12}$$

$$= \frac{1120 \times 10^2}{12} \times 12 = 112{,}000 \text{ lbs.-ins.}$$

$$\text{Shear} \quad = \frac{1120 \times 10}{2} = 5600 \text{ lbs.}$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{112{,}000}{138 \times 12}} = 8{\cdot}23 \text{ ins.}$$

say 10-in. slab with $d = 8{\cdot}5$ ins.

$$A_s = \frac{112{,}000}{18{,}000 \times 0{\cdot}857 \times 8{\cdot}5} = 0{\cdot}852 \text{ sq. ins.}$$

say $\frac{3}{4}$-in. bars every 6 ins. (0·88).

$b\,j\,d$ for shear $= 12 \times 0{\cdot}857 \times 8{\cdot}5 = 87$ sq. ins.

$$\text{Shear stress} = \frac{5600}{87} = 65 \text{ lbs. per sq. in.}$$

Concrete $\frac{10}{36}$	0·278 cu. yds.
Steel $0{\cdot}88 \times 40 + 5$ (say)	40 lbs.
Centering	edges only.

A floor carries $2\frac{1}{2}$ cwts. superload. It has many spans of 8 ft. 0 ins. and one span in the middle of 16 ft. 0 ins. Design this span for stresses of 750 and 18,000 with $m = 18$.

We may use Table CB IX with the condition $E\,I = \text{constant} \times (\text{span})^3$ for we shall increase the depth of the one large span.

Referring to the table $k = 2$.

The maximum moment at mid-span for arrangement (*a*)

$$\text{is} \quad M_y = 0{\cdot}106\, w\, (k\, l)^2$$

assuming a self-weight of 120 lbs. per sq. ft. this gives

$$0{\cdot}106 \times (280 + 120) \times 16^2 \text{ lbs.-ft.}$$

$$\text{or} \qquad 130{,}000 \text{ lbs.-ins.}$$

This is not quite correct as the self weight is not live load but it indicates that a 10-in. slab will be required for the large span while a 5-in. or $4\frac{1}{2}$-in. slab will do for the 8 ft. 0 ins. spans.

Referring to the figure in Table CB IX we see that we have :—

(i) A live load of 280 lbs. per sq. ft. which we must employ in arrangements (*a*) and (*c*).

(ii) A dead load of 60 lbs. per sq. ft. which we must allow for over every span as arrangement (*b*).

(iii) An additional dead load of 60 lbs. per sq. ft. on the large span only.

Taking the live load only in case (*a*)

$$M_y = 0{\cdot}106 \times 280 \times 16^2 \times 12 \text{ lbs.-ins.}$$

By subtraction $M_x = 0{\cdot}019 \times 280 \times 16^2 \times 12$ lbs.-ins.

Using the equation on the left of the table

$$4\,M_s + M_x = \frac{280 \times 8^2}{4} \times 12 \text{ lbs.-ins.}$$

These give

$$\begin{aligned} M_y &= 91{,}000 \text{ lbs.-ins.} \\ M_x &= 16{,}000 \quad ,, \\ M_s &= 9600 \quad ,, \end{aligned}$$

We can therefore draw a bending moment diagram for this case for the large span and one adjoining span for loading (*a*)

Passing to arrangement (*c*) we have

$$\begin{aligned} M_x &= 0{\cdot}048\, w\, (k\,l)^2 \\ &= 0{\cdot}048 \times 280 \times 16^2 \times 12 = 41{,}500 \text{ lbs.-ins.} \end{aligned}$$

also $$4\,\mathrm{M}_s + M_x = \frac{280 \times 8^2}{4} \times 12$$

$$M_s = 3100 \text{ lbs.-ins.}$$

We can draw a moment diagram for loading (*c*).

Similarly we may draw a moment diagram for loading (*b*).

We can now divide the large span into 8 parts and the small span into 4 parts and find the worst possible moment at each section by choosing the worst conditions from the three loadings (*a*), (*b*) and (*c*). These diagrams are not given here but the reader should draw them.

We now have an envelope of moment and can design the sections.

At mid-span of the large span under worst conditions we have

Moment due to live load only as case (*a*) . .	91,000 lbs.-ins.
Dead load of 60 lbs. per sq. ft. as case (*b*) . .	16,200 ,, ,,
Dead load of 60 lbs. per sq. ft. on large span alone (almost as case (*a*))	19,000 ,, ,,
Maximum positive M =	126,200 ,, ,,

$$\text{Minimum } d = \sqrt{\frac{126{,}200}{138 \times 12}} = 8{\cdot}75 \text{ ins. say 10-in. slab}$$

$$A_s = \frac{126{,}200}{18{,}000 \times 0{\cdot}857 \times 8{\cdot}8} = 0{\cdot}93 \text{ sq. ins.}$$

say ¾-in. every 5½ ins. (0·96).

The maximum moment at x is 51,800 lbs.-ins.

To resist this it is necessary to increase the thickness of the adjoining span by a splay. The moment soon dies away.

A detail of the suggested construction is in Fig. 127.

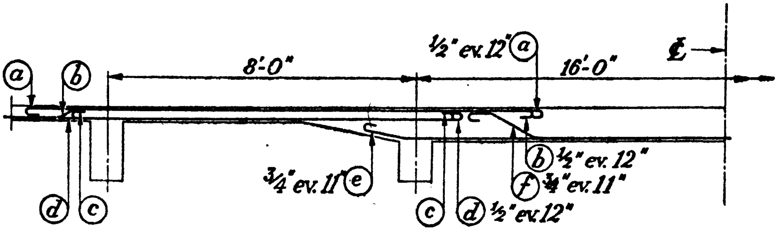


Fig. 127

This example shows the large amount of extra work involved by unequal spans. In addition to the working given above it has been necessary to sketch out three bending moment diagrams and one moment envelope. It is also necessary to estimate (or guess) special steel factors for the unusual arrangement of bars. Happily cases of unequal spans in slabs are very rare.

A brace runs through a series of columns. It is 14 ft. 0 ins. long in each span and carries a moment of 500,000 lbs.-ins. at the column face with a shear of 6000 lbs. Design one span using stresses of 750 and 18,000 with $m = 18$.

For an exposed brace with two rows of bars as (*b*) in Fig. 121 we shall need a width of 10 ins.

Aiming at a value of 200 bd^2 we have :—

$$\text{approx. } d = \sqrt{\frac{500{,}000}{200 \times 10}} = 15{\cdot}7 \text{ ins.}$$

Use a section 10 ins. × 18 ins. deep with $d = 16$ ins.

$$\text{Actual} \quad R = \frac{500{,}000}{10 \times 16^2} = 195$$

Steel required (see Table RB VII) is $A_s' = A'_s = 1\tfrac{1}{4}\%$.

$$1\tfrac{1}{4}\% = 1{\cdot}25 \times 10 \times 16 = 2{\cdot}0 \text{ sq. ins.}$$

Use 2 bars ⅞-in. lapped to give 4 bars ⅞-in. top and bottom (2·40).

$$b\,j\,d \text{ area} = \text{about } 10 \times 13{\cdot}7 = 137 \text{ sq. ins.}$$

Safe shear strength at 75 $b\,j\,d$ = 10,300 lbs.

No shear steel is required.

$$\text{Min}^{\text{m}}\text{. stirrups say } \frac{360 \times 10}{18{,}000} = 0{\cdot}2 \text{ sq. ins. per ft.}$$

(See Chapter XIV for this value.)

Using ¼-in. stirrups every 6 in. with two legs = 0·2 sq. ins. (from Table B II)

A detail is given in Fig. 128.

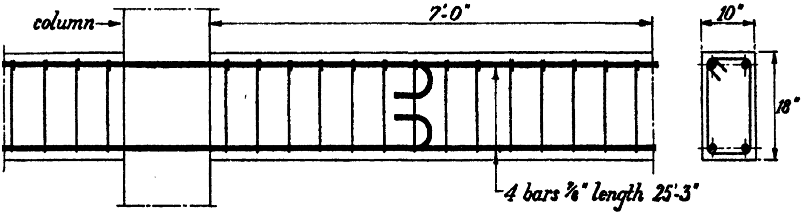


Fig. 128.

CHAPTER XIV

T-BEAMS

Size.—The most usual occurrence of T-beams is in slab and girder floors where the slab forms the flange of all the beams. This being so, it must be settled how much of the slab may be apportioned to each beam. This question is dependent on the shearing stresses set up. We shall first of all study the case of a secondary beam, i.e., one which supports a slab spanning one way, the span of the slab being at right angles to the span of the beam. In the case of main beams the slab itself has compressive stresses due to its own bending which are in the same direction as the compressive stresses caused by the T-beam bending.

The width of slab that can be taken as the flange of a main beam is therefore limited by the sum of the stresses set up by the local and main-beam bendings.

Analysis.—Let Figure 129 represent a plan and section of a T-beam.

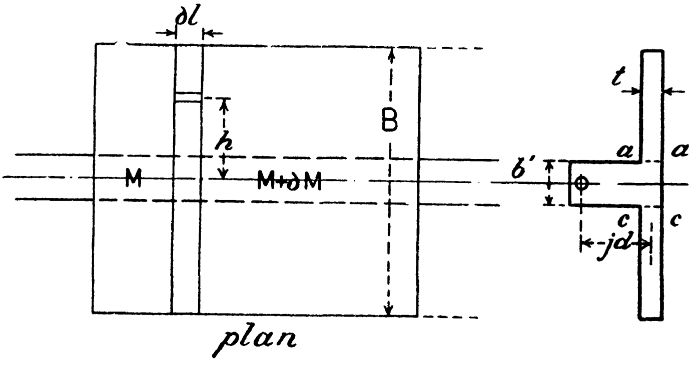


Fig. 129.

As we have already pointed out the flange is ineffective for reversed moments, being in tension. The beam therefore will have a rectangular section up to the points of contraflexure and a T-section between these points

Let M = bending moment

B = breadth of flange (assumed symmetrical).

(The differentials have been written $\delta\, l$, etc., in order to avoid confusion as the symbol d is always applied to the depth of the beam. The reader will understand that partial differentials are not intended.)

Let $$j\, d = \text{lever arm}$$

$$\text{Total compression in flange } = \frac{M}{j\, d}$$

$$\text{compression per unit width } = \frac{M}{B\, j\, d}.$$

Take two sections whose distance apart is $\delta\, l$ the bending moments being M and $M + \delta\, M$ and take a short length of flange $\delta\, h$ which is distant h from the centre line of the beam.

The compression on this short length of flange is

$\frac{M}{B\, j\, d}\, \delta\, h$ on one side and $\frac{M + \delta\, M}{B\, j\, d}\, \delta\, h$ on the other

the unbalanced thrust being $\frac{\delta\, M}{B\, j\, d}\, \delta\, h$ which is taken up by shear.

This horizontal shear is obviously zero when $h = \frac{B}{2}$ at the free edge of the slab and increases uniformly to a maximum value when $h = \frac{b'}{2}$ at the junction of the flange with the rib on section $a - a$ and $c - c$ in the figure. The area on which this shear acts between the two sections chosen is $t\, \delta\, l$. Therefore we have by integrating the unbalanced thrust from $h = \frac{B}{2}$ to $h = \frac{b'}{2}$.

Total unbalanced thrust between the two sections

$$= \frac{\delta\, M}{B\, j\, d}\left(\frac{B}{2} - \frac{b'}{2}\right).$$

Intensity of shear on section $a - a$

$$= \frac{\delta\, M}{B\, j\, d}\, \frac{(B - b')}{2} \times \frac{1}{t\, \delta\, l}$$

$$= \frac{\delta\, M}{\delta\, l} \times \frac{1}{2\, t\, j\, d} \times \frac{(B - b')}{B} = \frac{F}{2\, t\, j\, d} \times \frac{(B - b')}{B} \quad . \quad . \quad (1)$$

We have assumed that the intensity of compression in the flange is uniform throughout the depth t. The compressive stress will actually vary somewhat which means that the intensity of shear on section $a - a$ will be slightly greater at the top of the flange than at the bottom.

As far as the analysis goes we see that the calculated shear force intensity depends on the total shear at the section. When the bending moment has a maximum positive value which calls for the maximum allowable value of B the shear is zero. Conversely when the shear is a maximum (at the points of support) the bending moment is zero for free supports or negative for continuous beams which cases call for no flange width and therefore $(B - b')$ is zero.

This calculated shear is not the only stress on the slab, for the shear due to the loading and bending of the slab itself is also a maximum at the sections $a - a$ and $c - c$. Further we must remember that a very thin slab having a long span between beams might not take up the necessary deflection and curvature to make the whole T-beam deflect together and obey the simple laws of bending which we have tacitly assumed. The theoretical shear force which we have worked out will vary with the distribution of loads and bending moments, being higher for point loads than distributed ones. An extremely rough calculation may help to demonstrate one point.

A simply supported T-beam carries a load W at its centre point. The flange thickness t is small compared with the lever arm $j\,d$. The maximum compressive stress is $f_c = 600$ lbs. per sq. in. and the maximum intensity of shear (assuming adequate cross steel is provided) where the slab joins the rib of the T-beam is 100 lbs. per sq. in.

The maximum bending moment and maximum shear force occur at the same place (the example was chosen to illustrate the worst case)

$$M = \frac{W\,l}{4} \text{ (l is the span in inches).}$$

$$\text{Total compression in flange} = \frac{M}{j\,d}$$

$$= \text{approximately } f_c\,B\,t$$

$$\therefore \quad f_c\,B\,t = \frac{M}{j\,d} \text{ or } \frac{f_c}{M} = \frac{1}{B\,t\,j\,d}.$$

$$\text{Total shear} = \frac{W}{2} = F.$$

Writing these values in equation (1) above

$$\text{Intensity of shear} = \frac{F}{2\,t\,j\,d} \times \frac{(B - b')}{B}$$

$$= \frac{W}{2} \times \frac{1}{2} \times \frac{f_c}{M} \times (B - b')$$

$$= 100 \text{ lbs. per sq. in.}$$

also

$$M = \frac{W\,l}{4}$$

$$\therefore \quad 100 = \frac{W}{4} \times f_c \times \frac{4}{W\,l} \times (B - b')$$

$$(B - b') = \frac{100\,l}{f_c} = \frac{l}{6}$$

i.e., if t is reduced to its minimum value to resist the bending moment, in this particular case $(B - b')$ must not exceed one-sixth the span.

Lastly it is obvious that the width of flange taken for each beam in a row of beams cannot exceed the spacing centre to centre.

Width of Flange.—Rules for Secondaries.—All the foregoing is only approximate when applied to an actual structure. Practice shows that the following widths of flange should not be exceeded. The analysis given above will indicate the general reasons for each rule although it cannot give the exact numerical results. The flange width *B* should not exceed :—

(1) one-third of the span,
(2) the spacing of secondaries centre to centre,
(3) (twelve times the slab thickness) + (the rib thickness).

All the above shall only be valid if there is a continuous band of steel in the slab running at right angles to the line of the beam.

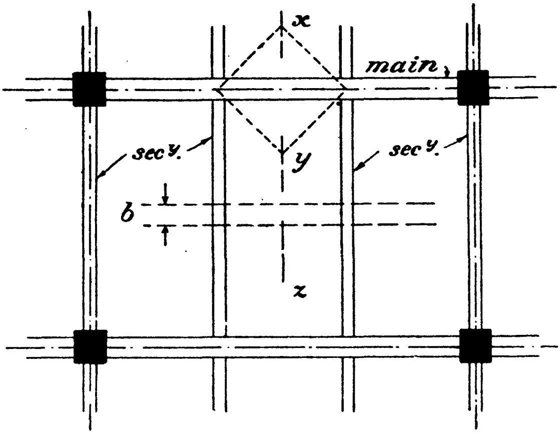


Fig. 130.

Width of Flange.—Main Beams.—In a slab and girder floor such as that illustrated in Fig. 130 we see that the floor slab spans the same way as the main beams which might lead us to suppose that the main beams have no effective flange width. This raises a point which we have so far avoided—namely, the conditions at the ends of a panel of floor. In Chapter XIII we have assumed that the length of panel was great in proportion to the span so that the end effects could be ignored. A discussion of transverse bending is given in Chapters XVII and XXIV.

The figure is a plan of the floor looking up (quite a usual way of drawing reinforced-concrete floors) and shows a main beam carrying two secondaries. The floor slab spans from secondary to secondary. Taking a width of slab *b* we see that when the main beam is fully loaded this width of slab is fully loaded. On a section *x y z* we shall have a positive moment in the slab and in the main beam both of which cause compressive stresses at the top of the floor slab.

If, however, we were to choose our width *b* near to the main beam the transverse stiffness of the slab resting on top of the main beam would prevent any considerable deflection and thus prevent any bending stresses. From experiments carried out on models the author concludes that the effective reduction of flange width of the main beam in such a case is much less than might be supposed at first glance. The maximum moments occur on diagonal lines at 45° to the main beam and the bending stresses in the slab on section $x - y$ within these diagonal lines shown in Fig. 130 are small. Cases of square slab bending are illustrated in Fig. 131. In these cases the maximum positive moments occur on the lines indicated and it can be seen that in case I a value of one-third the span for both beams is justified. In case II a value of one-third the span is

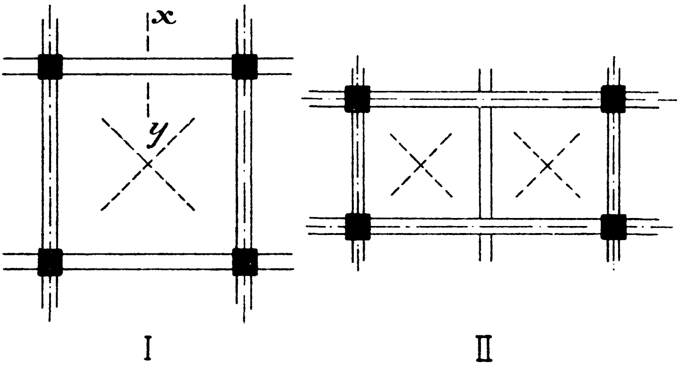


Fig. 131.

safe for the secondary and as the flange of the main beam is practically equal in width to the flange of the secondary by inspection an equal value may be taken. The reader should draw out a few further cases himself and by drawing the lines of maximum moment in the slab decide on a fair value of B for the main beams.

The flange width B for a main beam should not exceed

(1) one-third of the span,

(2) the allowable value of B for the secondary beams framing into it.

Provided always that transverse steel is provided that shall run across the whole width of the flange.

Width of Flange.—Isolated T-beams.—In the cases we have already discussed the floor slab has been supported on beams at both ends of the spans. In isolated beams which sometimes occur as gangways, etc., and in footing beams the slab cantilevers off the beam. If there is any chance of eccentric loading causing excessive twisting action on the whole system then this form of construction must be avoided.

The flange-width B for an isolated beam should not exceed :—

(1) one-quarter the span,

(2) six times the flange thickness,

in addition to which the rib of the T-beam should have a thickness b' of not less than one-third the overall depth and should have at least two vertical rows of bars side by side.

Width of Flange ⅂-beams.—The general considerations for ⅂-beams are similar to those for T-beams with the addition that the rib of the beam is not placed centrally under the middle of the flange. Any torsional effect produced by this state of affairs must be resisted by the floor slab or the beams which frame into it. Isolated ⅂-beams are never used.

The flange width B for a ⅂-beam should not exceed

(1) (one-half the allowable width for a T-beam in similar circumstances) plus (half the width of the rib of the beam).

Proportions of Rib.—The ratio of depth to span will vary from about one-twentieth for light loads such as roof loads, to one-fifteenth for moderate floor loads and one-twelfth for heavy floor loads (say 2½ cwts. per sq. ft. and over). Footing beams and beams in raft foundations where the loads are in the nature of one ton per square foot may be as deep as one-third the span.

The ratio of breadth to overall depth may be taken as varying from one-third to one-half being larger for heavier loads, usually on account of headroom. Footing and raft beams where depth of construction is no object may be constructed so that the width of the projection is one-third of its depth.

The total rib area being very nearly equal to the $b\,j\,d$ area must be sufficient to carry the shear (see Chapter IX).

The above proportions are, of course, only average values as special conditions may necessitate specially shallow or specially deep construction in certain cases.

Common Sections.—Most engineers of the author's acquaintance (including the author himself) have attempted to lay down standard sections for T-beams such that a section might be found by referring to tabulated values in the same way as one picks out a steel joist or a compound from the standard sizes by referring to the tabulated values of the moduli. Some have even gone to the length of working out standard bays of slab and girder floors. All these attempts have proved unsuccessful. It is one of the advantages of reinforced concrete that sections are capable of a wide variation and may be suited to any case. For example suppose we have a list of standard sizes ranging from a bending strength of 200,000 lbs.-ins. to 5,000,000 lbs.-ins. and that there are eleven standard sizes all told so that each represents an increase in strength of about 38%, over the section next below it in the table. If we design a series of beams by picking out the next size larger in the table our bending strengths will average about 20% above what is required, entailing an unwarrantable waste of material.

It is found to be by far the best plan to arrange the bars in vertical rows at the mid-span section, very small beams having one row of bars, moderate beams having two rows of bars and large beams having three rows. In order to allow the bars to pass at the supports at least 4 ins. per row must be allowed for bars up to 1⅛ in. diameter. Some beginners find difficulty in following the reasons for such an arrangement as the widths appear excessive while the loss of effective depth by piling bars one on top of another seems wasteful. A study of the sections at the supports and the general bar arrangement in beams will soon show the reasons for its adoption.

The sizes given in Table TB II are therefore not to be used as standard sizes in the sense of being slavishly followed and incorporated into a design as they stand. They are put forward as being fair average sizes for floor work and are useful as indicating the approximate size in any special case.

The question as to whether the bars should be allowed to rest directly on top of one another, being actually in contact, or whether they should be kept separate is a matter of argument. Some designers specify that at least ½ in. clear must be left between bars arranged in vertical rows this being accomplished in construction by inserting short pieces of ½-in. bar to act as spacers between the layers of bars.

In the author's opinion the actual arrangement of bars at the mid-span section itself is of very little importance no harm being done if the bars are in contact. It is however absolutely essential that the arrangement adopted must give a sound workmanlike and easily concretable section at the supports where the bars lap. The weakest point of modern slab and girder floors is the shear resistance of the T-beams particularly where the secondaries frame into the outside bressummers and where the mains frame into the outside columns. The only way of ensuring strength at these points is to supply an ample width of concrete which allows of running all the bars through without undue crowding.

This calls for a minimum width of 4 ins. for each vertical row for moderate-sized bars and more for larger bars.

Percentage of Steel. Number of Bars.—For purposes of calculation at mid-span we should take the percentage area calculated as $\frac{A_s}{Bd}$. This however is not the deciding factor. The main point is the percentage area of steel that occurs at the support sections when the beam becomes rectangular. The more steel that is crowded into the support section the more difficult it is to ensure sound concreting and the higher the local bond and shearing stresses. However much main steel or shear steel is supplied nothing will compensate for the loss of continuity caused by air pockets under and between the bars or by the occurrence of bundles of bars huddled together which practically divide the concrete section into two or three separate parts. To state an extreme case. Analysis will show that a beam having 10% of steel top and bottom will develop a safe bending strength of more than 500 bd^2. In practice such a beam would consist of a bundle of steel bars covered with a thin layer of cement grout and would be unable to act together as a coherent whole. If concreted under the conditions which usually obtain on a practical job its safe strength would be zero.

The percentage area of steel at the support should be limited to 3% top and bottom, i.e., a total of 6%. When choosing a mid-span section therefore we must bear in mind what section will result at the supports. The general aspect of the correct number and size of bars to be employed has been discussed in Chapter XI. In T-beams the most convenient number is four for beams up to a bending strength of about 1,250,000 lbs.-ins. One bar in each vertical row can then be bent up.

For longer spans more bars are desirable.

Secondaries and mains are discussed separately.

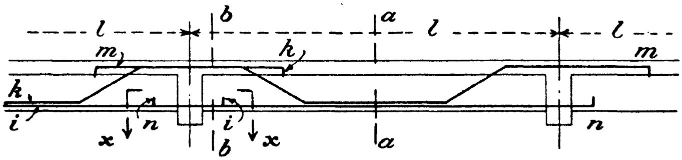


Fig. 132.

Secondary Beams.—Interior Spans.—These have been given a separate place in the discussion as they represent a simpler case. They are usually of shorter span than the main beams and, being carried by other beams, are never haunched at points of support. This means that the number of bars (in each vertical row) will be less and that the relation between the mid-span section and the resulting support section is simpler. We come now to the crucial point—the arrangement of bars and will consider first of all the case of equal spans.

In Fig. 132 an arrangement of bars similar to the arrangement of floor-slab bars is shown namely one bar such as $m - m$ in each bay being bent up and one bar such as $n - n$ being straight.

Assuming that these two bars are the same size, which is the case in standard design we see that the section $b - b$ which results from this arrangement has the same area of steel top and bottom and that this area is equal to the area at mid-span. The maximum moments we are designing for being $\frac{wl^2}{12}$ at centre and supports (secondary beams usually carry a uniformly distributed load) it appears that if we adopt the Steel Beam Theory (see Chapter III) such an arrangement is ideal. On this theory the lever-arm depth at mid-span and at supports may be taken as identical and as the safe bending strength is given by :—

$$18{,}000 \times A_s \times j\,d \text{ in both cases}$$

an area A_s which will suffice at mid-span will also suffice at the supports. Figure 133 shows the two sections.

Comparing the two sections we see that the bars *m* and *n* occur in the vertical centre line of the section at mid-span and not at the support as they are taken to one side to allow bars *i* and *k* to pass them. In the case of the bent-up bar *m* no difficulty occurs as the bent-up ends can be rocked to one side. The method of passing the bottom bars varies. Some designers actually bend the bars in plan as shown in Fig. 134, which represents plan *x* — *x* in Fig. 132. Such a bend should be gradual, a sharp kink being avoided.

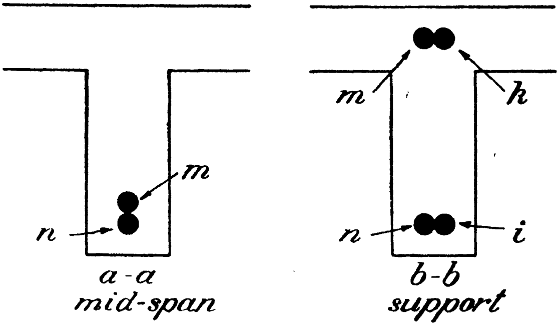


Fig. 133.

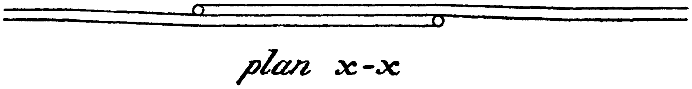


Fig. 134.

Some designers bend one bar up over the other which is fairly simple with fish-tailed ends but awkward when hooks are employed. In some cases the bottom bars are left straight and dropped into the boxes to take up their own position. The length of the bottom bars should be such that they project a distance of 20 diameters or one-tenth the span past the support at each end. It may be argued that this value of 20 diameters is not sufficient to justify employment of the Steel Beam Theory which assumes a stress of 18,000 lbs. per sq. in. in the steel and indicates a minimum distance (since both bars must be effective) of 45 diameters past the support for both bars. There is no doubt that the Steel Beam Theory is wrong in principle and a shorter lap will suffice. As pointed out in Chapter III the percentage of steel top and bottom should not exceed about 2·5%.

An elevation of the bent-up bar is in Fig. 135. In theory the distance *x* should not be less than about 0·1 *l* and *y* not more than

$0{\cdot}2\,l$ to cover the bending moment envelope (see Tables CB V and CB VII). In practice it is more important to make the bar effective in shear and for this the top bend should lie inside the 45° line. Some designers make the distance u equal to one-third the span. Although not in accordance with academic theory this has much to recommend it practically. The angle θ should preferably be not more than 30° for secondaries.

Although many existing beams have been designed with only two bars at mid-span as in Fig. 133, this arrangement is now seldom used, even the smallest beams having four bars as in Fig. 136 the elevation of the bars remaining as in Fig. 132.

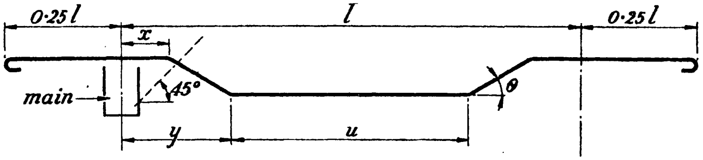


Fig. 135.

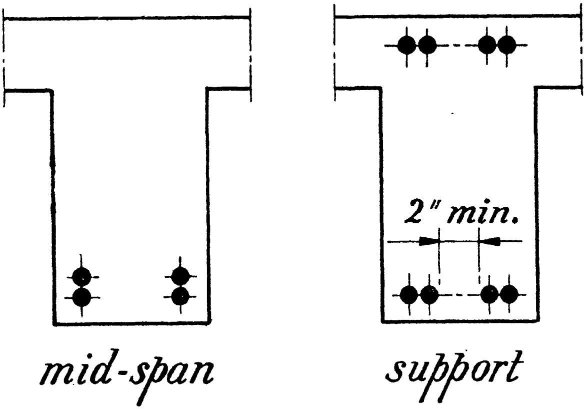


Fig. 136.

Shear in Secondaries.—It appears from Table CB VI that we can calculate the shear forces as for a simply supported beam. The maximum shear force at mid-span is therefore $\pm \frac{wl}{8}$ due to live load, and zero due to dead load.

At the support section we have a total shear force of $\frac{wl}{2}$ for both live and dead loads.

As far as intermediate points are concerned we may set up these two values and join them with a straight line without introducing much error.

To resist this we have already :—

either $75\ b'\ j\ d$ for the concrete

or $18{,}000 \times \sin\theta \times \frac{A_s}{2}$ for the bars.

If the total shear exceeds both of these then stirrups must be added to take the excess. In any case a few stirrups will be added as ties.

The type of stirrup employed will be as shown in Fig. 69, examples (*a*) and (*b*). They will be bent over the top of the bent-up bars for a distance of one-quarter of the span on either side of the support, but will require a special top bar for the centre half of the span. Top bars in secondaries should be ½-in. diameter.

Secondary Beams.—End Spans.—The easiest way of dealing with end spans is, as indicated for the case of slabs, by making them a trifle shorter than the interior spans. If this is not practicable we must increase the area of steel. It is not advisable to change the concrete section and in some cases, where the secondaries are deep and the mains shallow, an increase in depth would be impossible. A suitable mid-span section for the end span is easily found from the mid-span section for interior spans by increasing the area of tensile steel in the ratio $\frac{12}{10}$. If we do this we shall find that the steel at the support next to the end, being the average of the steel for the interior and end spans, is only about $\frac{11}{10}$ of the steel for the interior spans whereas by the Steel Beam Theory it should be $\frac{12}{10}$ as it has to resist a bending moment of $\frac{w\,l^2}{10}$. The necessary area may be obtained either by increasing the bars in the end span still further up to a value of about $\frac{14}{10}$ of the interior spans or by adding short extra bars. The former method is undoubtedly the sounder construction while the latter requires less total weight of steel. The free end of the beam requires particular attention both top and bottom bars being taken as far into the outside bressummer as possible and provided with good hooks. The bars in the slab will pass over the bars in the secondaries but if the cover to the bars in the beams is 1 in. and the cover to the bars in the slab is only ½ in., no special provision is required unless the slab bars are larger than ½-in. diameter, which is unusual.

Main Beams.—In general these have more bars and a higher percentage of steel on the *b'd* area at mid-span. Where this percentage exceeds about 3% they should be increased in depth at the columns by splays or brackets. The addition of these splays must disturb the distribution of bending moments, and in consequence, when they are provided, the section at the support should be checked by the usual theory of rectangular sections, no extension to the Steel Beam Theory being allowed.

In a main beam the shear is practically constant from the column to the first secondary beam and unless the arrangement of bent-up bars covers this length it is not really satisfactory as shear reinforcement. The difficulty usually lies in finding an arrangement that is really efficient in shear and yet which comes reasonably near to covering the bending moment envelope. In choosing between two possible arrangements the one which is stronger in shear should always be adopted. However the reinforcement is arranged it should be symmetrical at all cross-sections of the beam.

For example in a beam having six bars at mid-span arranged in three vertical rows of two bars each, the upper bars in the outer rows might be bent up near the support and the upper bar in the middle row bent up nearer the centre of the span. It would be unusual to bend up the upper bar in one outer row at one place and the upper bar in the other outer row at another place as this would not be symmetrical. The final choice of a mid-span section depends therefore on

(1) The ratio of the maximum moments to be resisted at mid-span and at support.

(2) The possibility of increasing the section at the support by splays or haunches.

(3) The total shear and particularly the length of span over which the bent-up bars will be called on to carry this shear.

(4) The thickness of the floor slab and the width of this which is available as flange.

There are one or two minor points. The depth of the main beams must be greater than the depth of the secondaries as the secondary steel has to pass over the main steel, and the width of the column must be sufficient to allow the column bars to pass the beam bars. (This is a point in favour of the square column with only four bars as these can be taken up outside all the beam bars without any trouble.) These two points as a rule arrange themselves and need no special provision.

Dealing with the first of our main points it is as well to examine the effect of different spacings of secondaries on the bending moments and shears in the main beam. Referring to Tables CB III, CB IV and CB V we have the theoretical maxima of moments for spacings of $\frac{1}{2}$ and $\frac{1}{3}$ the span and for a uniform distribution which we may take as applying to closer spacings. In a concrete floor the width of the beams and depth of construction both tend to spread any point loads and consequently any spacing closer than one-third the span may be treated as a distributed load.

In Fig. 137 we have a floor with the columns spaced s one way centre to centre and l in the other direction such that the floor slab spans $\frac{l}{2}$, the secondaries span s while the main beams span l. The

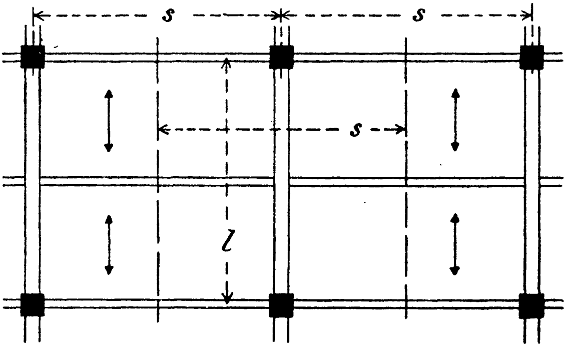


Fig. 137.

total load carried by each secondary is $\frac{l}{2} \times s \times w$ where w is the combined dead and superload. The reaction of the secondary beam on the main beam is $\frac{s\,w\,l}{4}$ (neglecting the self-weight of the beams for the time being) and consequently the total load carried by the main beam at mid-span is $\frac{s\,w\,l}{2}$.

If the secondaries were spaced at one-third the span the total loads at the third points would be $\frac{s\,w\,l}{3}$.

If the secondaries were spaced more closely we could regard the loading on the main beam as uniformly distributed and equal to $w\,s$ per unit length.

Using these results in Tables CB III, IV and V and modifying them as shown by dotted lines to agree with Table CB VIII for centre spans we should have the following results.

Table CB III

$$\text{Mid-span moment} = \cdot 175\,W\,l = \frac{\cdot 175\,s\,w\,l^2}{2} = \cdot 0875\,s\,w\,l^2$$

$$\text{Support moment (negative)} = \cdot 140\,W\,l = \frac{\cdot 14\,s\,w\,l^2}{2} = 0{\cdot}070\,s\,w\,l^2$$

Table CB IV

$$\text{Mid-span moment} = \cdot 111\,W\,l = \frac{\cdot 111 \times 2\,s\,w\,l^2}{3} = \cdot 074\,s\,w\,l^2$$

$$\text{Support moment} = \cdot 120\,W\,l = \frac{\cdot 120 \times 2\,s\,w\,l^2}{3} = \cdot 080\,s\,w\,l^2$$

Table CB V as Table CB VIII

$$\text{mid-span} = \cdot 083\,s\,w\,l^2$$
$$\text{support} = \cdot 083\,s\,w\,l^2$$

These results which are surprisingly uniform, do not show very much total difference between the bending moments. The shear force, however, varies greatly :—

Table CB III Max^m. shear $= \frac{W}{2} = \frac{s\,w\,l}{4}$

Table CB IV Max^m. shear $= \frac{W}{2} = \frac{s\,w\,l}{3}$

Table CB V Max^m. shear $= \frac{s\,w\,l}{2}$.

(The value of *W* of course is different in Tables CB III and CB IV although the same symbol has been used. A reference to the table in question shows the value of this symbol in each case.)

In the case of a central point load the shear although smaller in total value occurs over a longer part of the span and is correspondingly more difficult to cope with. All things considered it appears that a spacing of one-third l is theoretically the most advantageous. In a practical case the spacing is decided mostly by the superload and the actual span of the main beams. These two points between them settle the thickness of the slab on which so much depends. It is usual, although not essential, to make the spacing of the secondary beams some fraction of the span of the main beams so that one secondary occurs at each column. In most slab-and-girder floors a span of 6 ft. 0 ins. to 9 ft. 0 ins. for the slab will be found most advantageous and 7 ft. 0 ins. to 8 ft. 0 ins. may be aimed at. This will be further discussed in the Design of Structures.

As far as the main beams are concerned we may take it that the maximum bending moments at centre and supports are numerically equal for the purposes of general discussion.

The possibility of increasing the depth of the beam at the columns is a question of headroom and must be decided separately for each particular case.

In finally deciding the thickness of the floor slab for any design it is as well to run out roughly the bending-moment on the main beams and see what flange they require. It is not possible always to provide the necessary flange but it is advisable to try.

We must now examine some of the usual arrangements of bars and splays.

Suppose we have a T-beam (one of a series) which has 4% of tensile steel at mid-span reckoned on $b' \times d$ (the width of the rib multiplied by the effective depth) and suppose that we bend up half the steel and carry the other half straight through, we shall have then a rectangular section at the supports having $\frac{4 \times b' \times d}{100}$ square inches of steel top and bottom. If no haunch is provided we have 4% top and bottom.

Taking the lever arm at mid-span as ·9 d we have :—

Bending strength at mid-span $= 18{,}000 \times \frac{4\,b'\,d}{100} \times \cdot 9\,d = 650\,b'\,d^2$
at the support (Table RB VII) $= 512\,b'\,d^2$

on the usual theory. Unless we are prepared to adopt the Steel-Beam Theory which is only applied to secondaries with less than $2\frac{1}{2}\%$ of steel we see that the support section without a haunch is less than 80% as strong as the mid-span section.

If we increase d 20% by haunching, still keeping to our original steel, we have :—

$$p_s = p_s' = \frac{4 \times 100}{1{\cdot}2\, d \times b'} = 3{\cdot}35\%$$

$$\text{Safe } M = 450 \times b' \times (1{\cdot}2\, d)^2 = 650\, b'\, d^2.$$

If we increase d by 40%

$$p_s' = p_s = 2{\cdot}85\%$$

$$\text{Safe } M = 400 \times b' \times (1{\cdot}4d)^2 = 785\, b'\, d^2$$

If we bend up two-thirds of the main steel and run one-third through in the bottom of the splay the bending strength at the support for an increase in depth of 40% is $685\, b'\, d^2$ and for an increase of 50% is $752\, b'\, d^2$. If we bend up two-thirds of the main steel, run one-third through horizontally (see Fig. 140) and provide separate steel in the splay equal to one-third of the main tensile steel the figures are $580\, b'\, d^2$ and $642\, b'\, d^2$ for increases of 40% and 50% respectively.

All the above are taken from Table RB VII. For other percentages of steel at mid-span, other values of f_c, f_s or m, or other arrangements of steel, the bending strengths will be somewhat different but in general the increase in depth at the column is 40% to 50%.

The extent of the haunch lengthways must now be decided. If we take the strength of the unhaunched section as 50% of the strength at the maximum depth we shall cover most cases. Referring to Table CB VIII to find the point in the beam where the negative moment is $\frac{wl^2}{24}$, we find it occurs about ·13 l from the support. The splay should die out therefore about one-sixth of the span from the column centre. For short beams with deep splays the angle of splay should not exceed 30°.

There is one alternative method of strengthening a beam which may be mentioned. This consists in increasing the steel at the supports by introducing extra bars. If the total percentage of steel including the extra bars does not exceed about 4% top and bottom and if these extra bars are taken at least one-quarter the span past the support on either side this method may be permitted. The practice of crowding a large number of short bars into a small beam web although it may show an increase of calculated strength will actually decrease the shear strength without any increase of moment of resistance.

If the maximum moments in a beam are caused by heavy rolling loads the support moment will in general be less than the mid-span moment, for a single heavy load at or near mid-span causes a moment of ·171 $W\, l$ while a single heavy load, however placed, can cause no more than ·085 $W\, l$ at the support. The case where two equal heavy loads may occur, one on each of two adjacent spans approximates more to the case of a continuous system of loading and could cause a theoretical support moment of $2 \times \cdot 085\, W\, l = \cdot 17\, W\, l$ equal to the maximum mid-span effect. (See Plates I and II.)

Arrangement of Bars (General Considerations).—In main beams the total intensity of shear approximates to the limiting total shear much more often than in secondary beams. Where splays are provided the section where the splay dies out is usually the critical section.

Suppose we are working to a limiting total shear of $225\, b'\, j\, d$ and suppose that our beam at the critical section is actually stressed up to this value. If we have at our critical section a single system of bent-up bars making an angle θ with the neutral axis and if the area of this system (which may be one, two or more parallel bars side by side) is A sq. ins., its strength in shear is $18{,}000\, A \sin \theta$.

Let us assume that $\theta = 30°$ and write :—

$$\text{Shear strength of system} = 18{,}000\, A \times \frac{1}{2} \times \frac{b'\, j\, d}{b'\, j\, d}$$

Putting $j\, d = 0{\cdot}9\, d$

$$\begin{aligned}\text{Shear strength} &= \left(18{,}000\, A \times \frac{1}{2} \times \frac{1}{b' \times (0{\cdot}9 d)}\right) \times b'\, j\, d \\ &= 100 \times \left(\frac{100\, A}{b'\, d}\right) \times b'\, j\, d\end{aligned}$$

Now $\dfrac{100\, A}{b'\, d}$ is the percentage area of the system calculated on the area $b'\, d$.

Thus if the beam has 2% area of tensile steel calculated on $b'\, d$ at mid-span and if one-half of this is bent up at the critical section at 30°

$$\frac{100\, A}{b'\, d} = 1$$

and shear strength of system $= 100\, b'\, j\, d$.

If the bars were bent up at 20° we should have a strength of $68\, b'\, j\, d$. If at 45° a strength of $141\, b'\, j\, d$.

If the beam had 4% at mid-span and one-third of this were bent up at the critical section at 45°, we should have a shear strength

$$\frac{4}{3} \times 141 = 188\, b'\, j\, d.$$

STRENGTH OF SINGLE SYSTEMS IN TERMS OF $b'\, j\, d$ IN LBS./INS.^{2}

	$\theta = 20°$			$\theta = 30°$			$\theta = 45°$		
Total A_s at mid-span	2%	3%	4%	2%	3%	4%	2%	3%	4%
½ A_s bent up . .	68	103	137	100	150	200	141	212	282
⅓ A_s bent up . .	46	68	91	67	100	133	94	141	187
¼ A_s bent up . .	34	52	68	50	75	100	70	106	141

The table above is merely an illustration of some usual values.

Coming next to the stirrups which we shall assume as vertical and well anchored and stressed to 18,000 lbs. per sq. in.

Strength of stirrup system (see Chapter IX).

$$= \frac{18{,}000 \times j\,d}{12} \times \text{(sectional area per foot of horizontal section)}$$

$$= \frac{18{,}000 \times b'\,j\,d}{12} \times \frac{12\ \text{(sectional area per foot of horizontal section)}}{12 \times b'}$$

$$= 18{,}000\ b'\,j\,d\ \text{(sectional area per sq. in. of horizontal section)}$$

$$= 180\ b'\,j\,d\ \text{(percentage area of horizontal section).}$$

Suppose we have a beam 8 ins. wide having $\frac{3}{8}$-in. stirrups at 7-in. centres, each stirrup having two legs in tension.

Area per foot of horizontal section $= 0{\cdot}19$ sq. ins. $\times\ 2$ which is $0{\cdot}38$ sq. ins. occurring in a section 8 ins. wide $\times$ 12 ins. long

$$= \frac{0{\cdot}38}{8 \times 12} \times 100 = 0{\cdot}395\%$$

The shear strength of this system assuming that the beam is 14 ins. deep at least, so that the stirrups are close enough together to prevent shear cracks opening between them

is $\quad 180\ b'\,j\,d \times 0{\cdot}395 = 71\ b'\,j\,d$

Shear Strengths of Two-Leg Stirrups on $b'\,j\,d$ area lbs. per sq. in.

DIAMETER	$\frac{3}{16}''$				$\frac{1}{4}''$				$\frac{3}{8}''$			
SPACING	4″	6″	8″	12″	4″	6″	8″	12″	4″	6″	8″	12″
$b' = 5''$	50	33	25	17	89	59	44	30	200	133	100	66
$= 6''$	42	28	21	14	74	49	37	25	166	110	83	55
$= 8''$	31	21	16	10	55	37	28	18	125	83	62	42
$= 10''$	25	17	12	8	44	30	22	15	100	66	50	33
$= 12''$	21	14	10	7	37	25	18	12	83	55	42	28

The strength of four-leg stirrups would be twice the tabulated values and beams of 12 in. width and upwards would most probably have them, beams of 8 ins. or less having two-leg stirrups only. It will be seen from the two tables that we can usually cover a value of 225 $b'\,j\,d$ lbs. by a combination of bent-up bars and a normal percentage of stirrups.

We may carry our discussion one stage further (always remembering that we are at this stage merely discussing general theoretical principles).

At mid-span we have

$$M = 18{,}000 \times A_s \times j\,d.$$

At the critical section we have

$$\text{Total Shear} \quad = F = 225\, b'\, j\, d$$

$$\frac{M}{F} = \frac{A_s \times 18{,}000}{225\, b'}$$

$$= \frac{(p_s \times b'\, d)}{100} \times \frac{18{,}000}{225\, b'}$$

$$\text{giving } d = \frac{100\, M}{80\, F\, p_s}$$

Taking the case of a beam under a superload w of span l and writing

$$M = \frac{wl^2}{12} \times 12 \text{ lbs.-ins. (where } l \text{ is in feet and } w \text{ in lbs.)}$$

$$F = \frac{wl}{2{\cdot}5} \text{(at critical section) approx.}$$

$$d \text{ (inches)} = \frac{100 \times wl^2}{80 \times wl} \times \frac{2{\cdot}5}{p_s} = \frac{l}{{\cdot}32\, p_s}$$

remembering that l is in feet we have

$$\text{depth} = \text{(approx.) } \frac{\text{span}}{4 \times p_s}$$

if $p_s = 4\%$, depth $= \frac{\text{span}}{16}$ (quite a usual practical value).

This is merely a theoretical relation but serves to show another reason for the underlying principles which govern the ratio of depth to span and determine the usual percentage of steel in T-beams when used as main beams and heavily stressed in shear.

Mid-span Shear.—We have so far discussed only the maximum shearing forces and must now examine the centre part of the beam where the minimum shears occur and where under any normal arrangement of steel no bent-up bars are to be found. Treating the beam as simply supported we see that the following central shears may be expected

(a) for a central point load W, shear $= \frac{W}{2}$

(b) for a beam loaded at the third points

shear $= \frac{1}{3} \times$ (live load contained in one point load).

This assumes that one of the third points carries dead load only, while the other carries the full live and dead load, an arrangement which is not usually quite possible in a slab-and-girder floor.

(c) For a floor carrying a distributed load the maximum central shear $= \frac{1}{4} \times \frac{1}{2} \times$ superload $\times$ span.

Excluding beams which have a central point load and beams which carry rolling loads it appears that the shear in the central part of the beam, say the middle third of the span, will not exceed one-quarter the maximum shear. This means that the intensity of shear is not more than $\frac{225}{4} = 56$ lbs. per sq. in. which can safely be taken on the concrete alone.

Minimum Area of Stirrups.—The function of stirrups in a beam is slightly different from that of the links in a column as regards their tying action on the compression bars. In a beam there is little likelihood of the concrete in the compression flange failing in shear as a plain concrete cube fails in a testing machine. On the other hand there are always local shears where bent-up bars are employed and stirrups must always be provided. The minimum may be taken as about 0·20% area of steel taken on a horizontal section at the neutral axis. This gives

one $\frac{1}{4}''$ double-leg stirrup at 10″ for $b' = 5''$
one $\frac{3}{8}''$,, ,, ,, ,, 14″ ,, $b' = 8''$
one $\frac{3}{8}''$,, ,, ,, ,, 11″ ,, $b' = 10''$.

For beams where the stirrups are small enough diameter to be placed as close as $\frac{d}{2}$ apart, this minimum percentage gives a shear strength of $36\,b\,j\,d$.

Stirrups have many practical uses apart from calculated shear strength. Particularly do they help in tying across the construction joint between flange and rib which always occurs in English practice.

Arrangement of Bars (application).—The general principles underlying the arrangement of shear bars is the most important item in T-beam design as failures show that the shear strength of the T-beams is the weakest part of a floor. These principles do not lend themselves to any exact analysis but their violation may lead to a design which, while showing a high calculated strength is yet actually weak. We may state them somewhat as follows :—

(I) The angle of slope for bent-up bars should not exceed 20° for shallow beams, 30° for ordinary floor beams and 45° for beams over a depth of one-twelfth the span.

(II) All bent-up bars must be carried past the " supports " if they are relied on for shear.

(III) All systems of shear steel must be as continuous as possible and vary uniformly (for example the shear system shown in Fig. 138 although in accordance with the analysis of Chapter IX is undoubtedly bad, owing to the sudden change from heavy stirrups to a bent bar and then back again to stirrups. The sharp bends in the bar might easily set up enough local shear to cause cracks in that part of the beam where no stirrups have been provided).

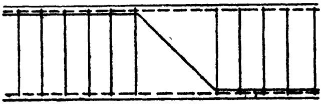


Fig. 138.

Some actual arrangements of steel are shown in Fig. 139.

The dotted lines at the top indicate top bars provided for the special purpose of hanging the stirrups to.

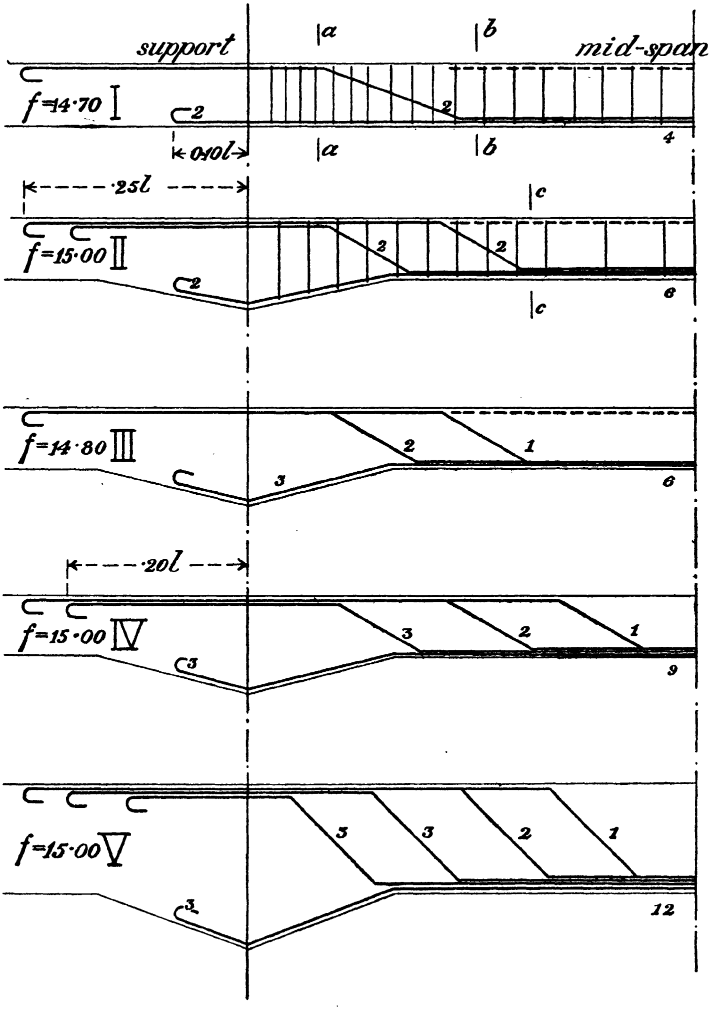


Fig. 139.

Arrangement No. I shows a light main beam having four bars at mid-span two being bent up at 20°. The top point of the bend is found as in Fig. 135. In this case a system of stirrups has been sketched in. This system must be able to take the excess shear at *b* — *b* and also at section *a* — *a* when reckoned in conjunction with the bent-up bars at the latter place. The direct support of the column may be assumed to extend as far as *a*—*a*. It might so happen that the two bent-up bars in conjunction with the concrete would take the whole shear at section *a*—*a*. In which case the stirrup system necessary at *b*—*b* would be continued right through to the support in accordance with our general principles although analysis would indicate that such a provision was unnecessary. Arrangement II shows a beam having six bars at mid-span arranged in two vertical rows. This has been sketched to give uniform shear strength from the support to section *c*—*c* as might be required for a beam loaded at the third points. Arrangement No. III has six bars at mid-span in three vertical rows. The middle upper bar is bent-up first and then the two outside upper bars giving greater shear strength

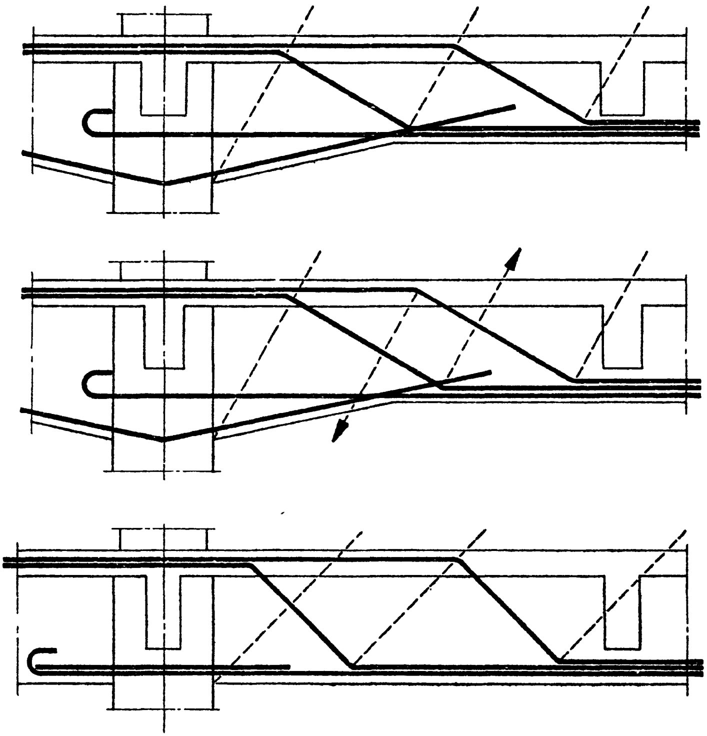


Fig. 140.

nearer the support. Arrangement No. IV has nine bars in three vertical rows. The top bar in the middle row is bent up first, then the two top bars in the outer rows, and finally the whole three middle bars of each row, the bottom bars being run through. Arrangement No. V shows a deep beam where the bars may be taken as a system. In all the other arrangements they can only be looked on as isolated bars (see Chapter IX).

Although it may seem most logical to fix the depth of the beam first and then arrange the steel in this depth it is sometimes advisable if the bar arrangement turns out very awkward in shear, to modify the overall depth of the beam to obtain better angles and better laps and then increase or decrease the steel accordingly. Fig. 140 shows three cases of a main beam loaded at the third points. The upper and lower figures give good shear arrangement and the bars bend up and reach the top flange at places reasonably in accordance with the bending moment envelope (Table CB IV). The middle beam has an awkward ratio of depth to span and would give neater arrangement if made a little shallower or a little deeper. (In the lower figure the depth is just over one-twelfth of the span thus justifying bending up at 45°, and the area of steel at mid-span is 2½% thus not requiring a splay at the support.)

Quantities in T-Beams.

(a) **Concrete.**—As the floor slab is taken as extending over the whole area of the floor, we are concerned with the concrete in the rib. This will be stated in cubic yards per yard run of beam. The volume occupied by the bars or cut off by small triangular chamfers is never deducted.

A beam *n* inches wide and *m* inches deep as in Fig. 141 requires

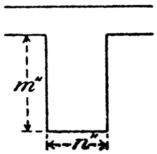


Fig. 141.

$$\frac{m \times n}{36^2} \text{ cub. yds. per yd. run.}$$

A beam 12 ins. wide by 20 ins. deep net requires

$$\frac{20 \times 12}{36^2} = 0{\cdot}185 \text{ cub. yds. per yd. run.}$$

In preliminary quantities all beams are measured centre to centre of columns or centre to centre of main beams with no deductions for intersections.

(b) **Centering.**—This will be taken as square yards per yard run of beam. In preliminary quantities a beam n inches wide and m inches deep as in Fig. 141 would require $\frac{2m + n}{36}$ sq. yds. per yd. run. This may seem illogical as the slab centering has already been taken as extending over the gross area of the floor which includes the width n. Beam boxes are more expensive than slab centering which is one argument for including n twice. The question is entirely one of custom. If a particular firm is in the habit of reckoning 2 m only in the beam centering and has analysed its previous jobs on this basis then the unit prices put in for the beam centering when estimates and tenders are being made up will be strictly in accordance with this method of taking off. All quantities therefore in this firm must be taken off allowing for 2 m only or the unit prices will not apply to the quantities calculated. The designer must discover what particular system is in vogue in the firm he designs for. In the case of a bill of quantities the method of taking off quantities should be clearly stated on the bill. The transition from one system of taking off units to another is simple and needs no explanation. In this work n will always be included in the beam centering.

Some designers use tabulated values showing the concrete and centering for different sizes but the author finds it easier to work them out by slide-rule as required.

(c) **Steel.**—In abnormal cases this may be estimated in three separate parts—main steel, top steel and stirrups. In most cases a combined factor will give fairly accurate results.

Suppose we have a beam 20 ft. 0 ins. span reinforced with bars 1 in. diam., the bars being arranged as I in Fig. 139. If the bottom bars are stressed in compression where they overlap at the support to about 10,000 lbs. per sq. in. they should lap about 25 diameters past the support or say 2 ft. 0 ins. These bars therefore will be 24 ft. 0 ins. long exclusive of end hooks. Using round hooks we shall have a bar say 25 ft. 6 ins. long total or 1·27 × 20′ 0″ that is 1·27 times the span. A bent-up bar which laps one quarter of the span will be 30 ft. 0 ins. long excluding end anchors and extra length for sloping pieces. If the beam is 20 ins. deep overall and the bars rise 17 ins. at a slope of 30° the sloping lengths will be 34 ins. long on the slope and 29½ ins. long in plan—a gain of 4½ ins. With round end-hooks the length of the bent-up bar will be

$$30' \, 0'' + 2 \times 4\tfrac{1}{2}'' + 2 \text{ hooks} = \text{say } 32' \, 3''$$

which is 1·61 × 20′ 0″ or 1·61 times the span.

Assuming the beam has the same number of bent bars as of straight bars the average length of bar is

$$\frac{1{\cdot}27 + 1{\cdot}61}{2} = 1{\cdot}44 \text{ times the span.}$$

If A_s is the area of steel at mid-span in square inches, the weight of steel to reinforce a 20ft. 0 in. span is

$$3{\cdot}4 \times A_s \times 20 \times 1{\cdot}44 \text{ lbs.}$$

$$20'\ 0'' = 6{\cdot}67 \text{ yds.}$$

$$\text{weight per yard} = \frac{3{\cdot}4\ A_s \times 20 \times 1{\cdot}44}{6{\cdot}67}$$

$$= 14{\cdot}7\ A_s$$

Our steel factor for this arrangement is therefore 14·7 for main bars only.

Various other arrangements are drawn in Fig. 139, the steel factor for each being given on the figure. On the average we may say a factor of 15 is about correct. (If separate bars are added to reinforce the splays their weight will be extra.)

Thus if a continuous beam has many equal spans all reinforced with four bars 1 in. diam. at mid-span the weight of main bars per yard of beam is very nearly 15 × 3·14 = 47 lbs. per yd.

For a single isolated span the factor for the main bars is about 11. For a beam of two spans only, it is about 13 or 13½.

Several points in bar arrangement arise from Fig. 139. The free top end of any bar should not be taken past the support further than the top angle of the bend-up in the layer immediately above it in the adjoining span. In the figure the bars in any one layer are kept wholly above the bars in the layer next below both in the span itself and in the adjoining span. This is the most convenient practical arrangement. The slope of the bend-ups should also be the same throughout the beam as this makes for cheap and uniform bending as the dimensions x and y in Fig. 142 will be constant for all bars.

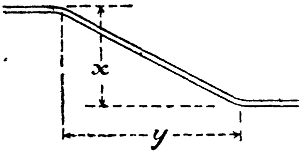


Fig. 142.

With equal spans there will be twice as many bars in each layer at the supports in Fig. 139 as there are at mid-span. Bars may be laid together in pairs at the supports but there must be at least 2 ins. horizontally between one pair and the next. (In the figures the bars have been drawn with sharp bends. These of course are never used in practice and are only diagrammatic.)

As a rule very deep beams contain a large amount of steel and it is worth while to run out a sketch of each particular case and take off the length of the bars roughly so as to get a closer estimate than may be obtained from comparison with a standard case.

It will be understood that these steel factors are only used for estimating the weight of steel in a preliminary design. When the job is finally detailed the exact weight can be obtained from the descriptive list. (For weights of bars in end spans see below.) The weight of the top bars depends on whether they are supplied merely to anchor the top of the stirrups or whether they are needed to help resist the compression. In the former case they will consist of two or three small bars—say ½-in. bars for small beams and ¾-in. bars for large beams. The bent-up bars from adjoining bays both sides project a distance of ·25 *l* into the span and these serve to carry the tops of the stirrups except for the middle half of the span. The length of these top bars, therefore, will be about ·6 *l* and their factor will be $3 \times 0{\cdot}6 \times 3{\cdot}4 = 6.$

The weight of top compression bars will vary with every case. Their use should be avoided as much as possible. When employed they should not be shorter than ·5 *l* and their factor will vary between 5 and 6, depending on the case.

The top bars in end spans when put in as stirrup carriers may be longer, depending on the arrangement of main steel. In Arrangement I in Fig. 139 if used for an end span the top bars (shown dotted) would have to be extended towards the free support until they met the top of the sloping part of the main bars. Arrangement No. III would also require longer top bars as the first bent-up bar is a single bar in the centre of the beam. Arrangements Nos. II, IV and V would require little modification. The top steel factors for Nos. I and III used as end spans will be about 7 for stirrup carriers. For compression bars which are needed to resist the bending moment the same factors will apply to the end spans as to the middle, namely, 5 to 6, depending on circumstances.

We have now to estimate the weight of stirrups. These, if light in section, may be evenly spaced throughout the span. If heavy stirrups are required at the point of maximum shear and not required at mid-span, then the spacing will be variable. In any case, one or two stirrups will be omitted where the beam is intersected by the column and by secondary beams (if any). The size of stirrups is calculated primarily as so many square inches per foot run of horizontal section and given in the bar tables as such. It is on this figure therefore that we must base our steel factor.

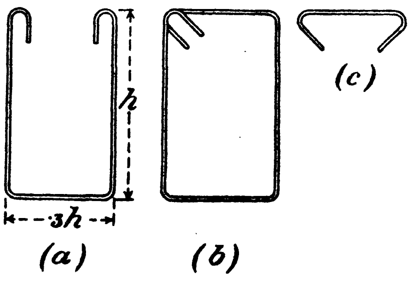


Fig. 143.

For open topped stirrups of type (*a*) in Fig. 143 we can assume the width as ·3 times the height *h*.

The length of bar required therefore $= 2{\cdot}3\,h +$ two short ends.

If we write :—

$$\text{length} = 2{\cdot}33 \times (\text{overall depth of beam})$$

we shall just about cover it and have a simpler expression. For type (*b*) we may write 2·6 × (overall depth).

Writing D = overall depth in feet

A = area of bar employed.

$$\text{Weight of one stirrup} = 2{\cdot}33\,D \times 3{\cdot}4\,A$$

$$\text{Effective area in shear} = 2\,A$$

$$\therefore \quad \frac{\text{Weight}}{\text{effective area}} = 3{\cdot}96\,D.$$

Extending this by ratio we have

$$\frac{\text{Weight per yard run of beam}}{\text{Area per foot of horizontal section}} = 12\,D.$$

For example if we have ⅜-in. stirrups at 12-in. centres we have an area of 2 × ·11 sq. ins. per foot of horizontal section at the neutral axis. If the beam were 18 ins. overall depth the weight per yard run of beam would be :—

$$12 \times \left(\frac{18}{12}\right) \times {\cdot}22 = 3{\cdot}95 \text{ lbs. per yard.}$$

Similarly for stirrups of type (*b*) (or also roughly for stirrups of type (*a*) combined with light cross ties over the top as (*c*) in Fig. 143) we have

$$\frac{\text{Weight per yard run of beam}}{\text{Area per foot of horizontal section}} = 13{\cdot}2\,D$$

(note—D is depth in feet).

For example ½-in. stirrups in pairs at 8-in. centres in a beam 30 ins. overall depth have an area of 4 × ·29 sq. ins. per foot (see bar tables). If made with closed tops they will have a weight of :—

$$13{\cdot}2 \times \left(\frac{30}{12}\right) \times 4 \times {\cdot}29 = 38{\cdot}5 \text{ lbs. per yard of beam.}$$

The weight of a system having variable spacing must be judged by circumstances. If the stirrups are very close together at the critical section in a deep beam say 3 ins. apart in a beam 30 ins. deep and if the shear force falls off rapidly, a much larger spacing, say 12-in. centres, may be employed in the middle half of the beam. In a beam loaded at the third points the stirrups may be evenly spaced in the two end thirds of the span and spaced out in the middle third. The average weight in such cases will vary between ·4 and 1·0 times the maximum weight. In an average floor a factor of 12 on the secondaries and 9 on the mains may be taken if the designer is in too great a hurry to sketch out any arrangements and make a rough count.

For normal floor beams without heavy top bars at mid-span and excluding extra bars (if any) added at the supports a total steel factor may be used.

Suppose we have a beam 20 ft. 0 ins. span 9 ins. × 16 ins. net in section under a 4-in. slab, reinforced with four bars 1 in. diam. at mid-span, two bars being straight and two bent up. Suppose this beam has many continuous spans of 20 ft. 0 ins. If the top bars are two bars ⅝-in. and the stirrups are two-leg ⅜-in. diam. at 8-in. centres we have

Area of main bars $A_s = 3{\cdot}14$ sq. ins.

Area of top bars 0·6 sq. ins.

Area of stirrups per 12 ins. length of neutral plane (two legs) $= 2 \times 0{\cdot}17 = 0{\cdot}34$ sq. ins.

Depth of beam = 20 ins. = 1·67 feet.

Weight of main bars per yard of beam $= 3{\cdot}14 \times 15 = 47$ lbs.
" " top " " " " " $= 0{\cdot}6 \times 6 = 3{\cdot}6$ "
" " stirrups $= 0{\cdot}34 \times 1{\cdot}67 \times 12 = 6{\cdot}8$ "

57·4

$$\text{Combined steel factor} = \frac{57{\cdot}4}{3{\cdot}14} = 18{\cdot}2$$

For all preliminary estimates the author uses a combined factor of 18 for fully continuous beams of normal design.

Thus if a continuous beam has 6 bars 1⅛-in. at mid-span (5·96 sq. ins.) the total weight of all steel is 5·96 × 18 say 108 lbs. per yd. run of beam, including stirrups and light top bars.

Extension to Rectangular Beams.—In rectangular sections which are loaded in one direction only the compression steel at mid-span may be looked upon as replacing the concrete flange in a T-beam. The tensile bars at mid-span are then arranged in some such manner as shown for the tensile bars in T-beams as in Fig. 139. This arrangement gives plenty of shear strength. The steel factors for the main bars, top bars and stirrups separately will therefore be the same. The method of calculating the concrete and centering is identical.

In rectangular beams which have to resist load in either direction (such for example as a beam in the dividing wall between two tanks either of which may be full while the other is empty) the outside layers of bars may be run through ·25 l past the points of support while the inside layers are bent up across to the opposite side of the beam and taken just past the support. Ample width must be allowed as the two bent-up layers cross one another. This suggested arrangement is shown in Fig. 144.

End Spans.—Unequal Spans.—The remarks already applied to secondaries will apply to main beams with this addition. If it is

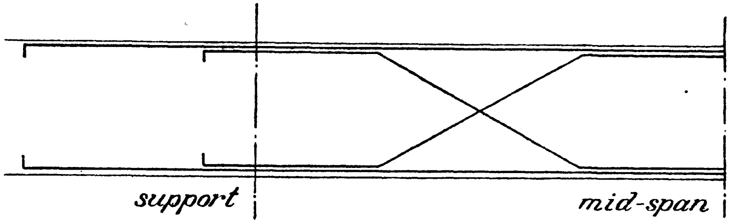


Fig. 144.

not found practicable to reduce the span of the end spans until their maximum moment is approximately the same as the centre spans, then we can make the splays at the support next the end support a little deeper than the splays elsewhere. For unequal spans we must use some such table as Table CB IX or direct calculation. The example given in Chapter XIII indicates the method and gives a good idea of how to guess a covering moment if a preliminary design is wanted in a hurry.

Γ-Beams.—These must always be provided with top steel at all points. This is required to help resist the twisting moment (most of which is resisted by the slab) caused by the load being applied eccentric to the top flange. It is as well to provide a fairly stout system of stirrups in addition. In most cases there is an additional reason for providing top steel in the fact that Γ-beams are subject to temperature stresses. The most frequent occurrence of such sections is as outside bressummers in buildings most of which have some system of internal heating. If the inside of a building is at 60° F. or more while the atmosphere outside is at 30° F. or less there will be a strong temperature gradient in the edge of the floor particularly in the bressummers. Outside beams which act as main beams should be made as wide as possible so that the bars of the secondary beams are provided with an anchorage to help the shear forces. Experience seems to show that this joint is one of the weakest points in floor construction.

CHAPTER XV

TIES

Sections.—The sections of ties have already been discussed and offer little difficulty, the main deciding factor being the conditions at the ends where the bars must be anchored.

Secondary Steel.—Where no shear has to be resisted some arbitrary size and spacing of links has to be adopted. Links, similar to column links, ¼-in. diameter spaced at a distance apart equal to the depth of the member may be used. These are always necessary (unless the steel consists of one bar only) as the bars are never laid quite straight and tend to straighten when loaded, thus tending to pull the concrete apart. If the tie has also to resist some transverse load, the section being under combined direct load and bending stress, the shear strength of the member will have to be checked (as for example in coal hoppers).

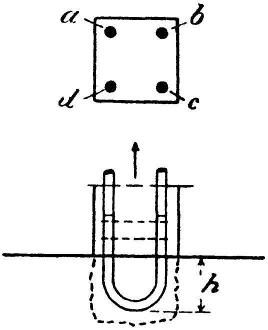


Fig. 145.

End Connections.—For light loads the bars may be bent round in a large hook, the ends being brought back again into the member. The conditions vary somewhat according to the arrangement of members at the joint.

Fig. 145 illustrates a case where a tie runs into another member at right angles, there being no other inclined members at the joint. The tie is shown having four bars *a, b, c* and *d.* Bar *a* is bent in a large hook the end of which is brought back and lies alongside bar *b.* Bar *b* is provided with a similar hook the end of which lies alongside bar *a*. Extra links are provided to prevent all the hooks opening when loaded. In this case where the bars are taken only a short distance into the transverse member there is a danger of the whole tie pulling out and fracture occurring on some such line as indicated in the figure. This action is somewhat akin to a reversed punching shear but considering the weakness of concrete in tension a limiting value of 75 lbs. per sq. in., calculated on an area of height *h* and length equal to the perimeter of the tie should not be exceeded.

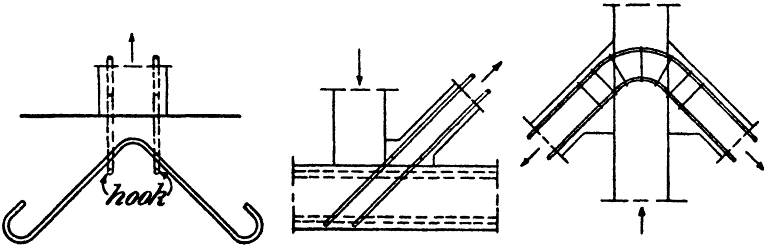


Fig. 146.

For example, a tie having four bars $\frac{3}{4}$-in. diam., carrying a tension of 28,200 lbs. and being 8 ins. × 8 ins. in section, should be taken in a depth *h* where

$$h \times (4 \times 8) \times 75 = 28{,}200$$

$$\text{i.e.,} \quad h = 12 \text{ ins.}$$

The strength of such a joint can be greatly increased by passing the bars in the tie round the main bars in the member into which it frames or by providing special anchor bars. These may be straight bars or of the shape shown in Fig. 146 in the left-hand sketch. This figure also illustrates two cases where the joint consists of more than two members, the right-hand sketch being a symmetrical case. The type of anchor bar shown will be found useful in cases where the ends of beams have to be suspended from slab walls, etc. For example in the case of a reservoir where the outer walls have vertical counterforts outside, these counterforts are supported top and bottom by being tied into the bottom slab and roofing slab of the reservoir. This suspension action may be obtained partly by curving round

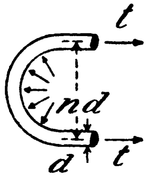


Fig. 147.

some of the main bars in the counterfort and running their ends well into the slab and partly by supplying anchor bars, these anchor bars passing through the hooks in the ends of other main bars of the counterfort.

The reader may have doubts about the bearing power of hooks in concrete. Suppose we have a semicircular end as shown in Fig. 147 each leg of the bar carrying a tension t, the diameter of the bar being d, the diameter of the bend being n times the bar diameter. The intensity of bearing pressure of the bar on the concrete round the bend is given by :—

$$\frac{2t}{nd \times d}.$$

If we put $t = 8000 \times \frac{\pi d^2}{4}$ which will correspond to the case in Fig. 145 (since the two legs have two hooks and we may work to a tensile stress of 16,000 lbs. per sq. in. in the steel) and make $n = 5$ diameters.

We shall have :—

$$\begin{aligned} \text{bearing pressure} &= 2 \times 8000 \times \frac{\pi d^2}{4} \times \frac{1}{5 \times d^2} \\ &= 800\,\pi \\ &= 2500 \text{ lbs. per sq. in.} \end{aligned}$$

This appears startlingly high at first sight since it exceeds the assumed ultimate strength of the concrete in compression. This case is, however, totally different from the case of a cube of concrete crushed in a testing machine. Isolated points of very high pressure may safely be allowed where a much lower stress intensity if spread uniformly over the whole area would be dangerous. The author knows of cases where pressures of 5 or 6 tons per sq. in. occur at isolated points in large masses of concrete, the lateral support of the surrounding mass of concrete preventing failure by crushing. On the other hand reliable data on the point are lacking. Anchor bars passing through the hook will serve to distribute the bearing pressure to some extent. The author thinks that a pressure of 2500 lbs. per sq. in. as obtained above is quite safe as far as compression goes. The really dangerous effect of such high local intensities is the local tensions and local shears set up which must be provided for by liberal transverse reinforcements at all points where large bars change direction suddenly. Such an intensity cannot be allowed unless the bend is well buried.

Anchor Plates.—Where large bars are employed and where grip length is very short, mild-steel anchor plates must be used. Such a plate is shown in Fig. 148, which also shows a case where such a plate may be required. The thickness t may be guessed or may be found by approximate calculation. If we look on the plate as being loaded uniformly on its bearing face and supported at the points where the rods are taken through, it approximates to the conditions of a flat-slab floor. Such a floor has a maximum moment of $\frac{wl^2}{16}$ or thereabout.

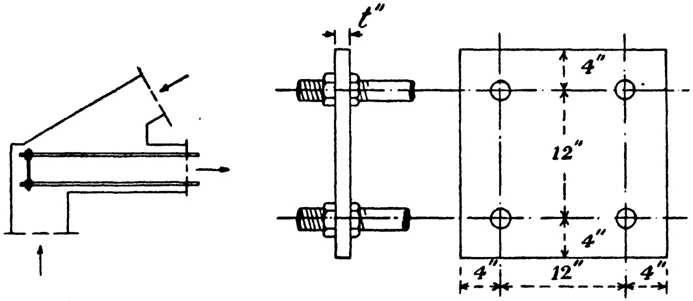


Fig. 148.

In the case illustrated we have a span of 12 ins. and may take $w = 600$ lbs. per sq. in. safe bearing. (If the area of plate is small compared with the overall section of the member we might exceed this.)

$$\text{Bending moment per inch width} = \frac{w\,l^2}{16}$$

$$= \frac{600 \times 12^2}{16} \text{ lbs.-ins.} = 2{\cdot}4 \text{ tons-ins.}$$

$$\text{Modulus} = \frac{1'' \times t^2}{6} \text{ for a strip 1 in. wide.}$$

Moment of Resistance at 7·5 tons per sq. in.

$$= \frac{t^2 \times 7{\cdot}5}{6} = 2{\cdot}4$$

giving $t = \sqrt{1{\cdot}91} = 1{\cdot}38 = 1^3/_8$ *ins.*

Considering all the circumstances this will be amply safe. If the bars are required to take their full tensile load right up to the end their area must be taken as the area at the bottom of the threads or the ends may be upset before screwing.

Very large ties may be composed of structural angles or flats, the end anchors being similar to the built-up base on a structural steel column.

CHAPTER XVI

COLUMNS AND STRUTS

End Conditions.—The realization of the reactions at the column junctions with the beams always presents great difficulty to the student. Suppose we have part of a perfect monolithic elastic structure as shown in Fig. 149.

If we apply a load to any member, this member must deflect causing the ends of the member to take up an inclination to its original line when unloaded. If the joint is perfectly rigid this inclination must be communicated to all the other members which meet at that joint. It appears therefore that a transverse load on any member will cause deflections and bending stresses in all the other members of the structure which lie in the same vertical plane. It is evident that the effect of this bending will die away rapidly in all directions, being a maximum in the immediate neighbourhood of the load. (Compare the bending moments on a continuous beam which die away in the ratio of 1 : 3·73 at successive points of support.)

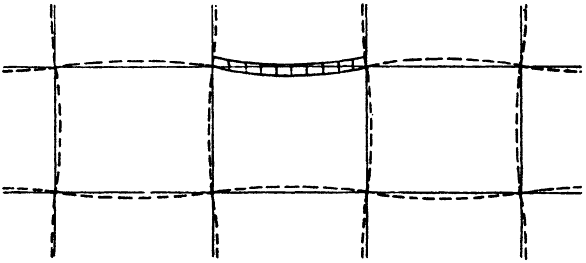


Fig. 149.

The total amount of elastic energy in the structure is, of course, equal to the work done by the load falling through a distance equal to the deflection of its centre of gravity. If we assume simple elastic conditions it is possible to calculate what the bending moments will be. In a reinforced concrete structure such calculation is neither necessary nor desirable. It has been the general practice for many years to design the columns in a building to carry the direct load alone and neglect all bending moment due to beam deflections, at the same time treating the beams as independent continuous beams and designing them for the full bending moment without deducting any restraint due to column stiffness. (Some few designers add a small amount of extra steel to the outside columns or design such columns for a small arbitrary eccentricity.)

Experience shows that such a course is sound and adequate. In a perfect elastic structure such bending moments would undoubtedly occur. A reinforced concrete structure under working loads behaves to some extent as an elastic continuous structure. Why, therefore, does it happen that these interactions should be neglected? The main reasons are as follow:—

(1) A reinforced concrete structure is structurally weak at sharp changes of section, particularly at construction joints.

(2) Stresses under working loads are no criterion of the ultimate strength of such a structure. The elastic conditions and elastic constants (Young's Modulus, the value of the moment of inertia, etc.) change rapidly as the load is increased thus destroying the original relations between the stiffnesses of the various members and causing a redistribution of moments. (Compare the end part of this chapter where long columns are discussed.)

The ultimate strength of a structure is the only safe measure of its load-carrying capacity. Therefore any distribution of bending moments which depends on the relative stiffness of members of different section, particularly where some carry direct loads, must be worked out on the stresses under ultimate loads. (It is a fact worthy of note that this point has escaped most writers on the subject although standard practice in column design has indicated such a necessity for years.)

(3) The assumption of points of contra-flexure and of zero moment in the columns where they intersect the beams, provided the beams are designed for free supports, is in accordance with the Principle of Greatest Strength and is therefore a fair assumption.

In conclusion it may be stated that millions of columns have been designed and constructed for direct load only. Such designs have proved invariably successful. It is futile to argue against the facts.

Were it necessary to take any account of bending due to beam deflections the matter could be met by some simple reduction in the working stresses in columns—say something akin to the rules which limit the breadth of a T-beam flange. The calculation of moments in an elastic structure and their application to a reinforced concrete building is too laborious a process to be adopted in a drawing office.

In slab-and-girder floors the columns will be designed for the direct load only. In flat-slab floors, where the stiffness of the columns is relied on to diminish the moments in the floor, the columns will be designed for an arbitrary moment. In cases where the columns are subject to actual transverse loads they must be designed for them.

For example, the legs of a water-tower under wind load must be designed in accordance with the remarks on braces. The assumption of points of zero moment where the columns intersect the braces would obviously result in the total collapse of the design.

Short Columns.—A column may be looked upon as short if its round-ended length does not exceed fifteen times its least outside diameter. Most reinforced concrete columns fulfil this condition and this case will be treated first.

Arrangement of Bars.—The arrangement of bars in columns is governed by the general principles in Chapter XI, the only difficulty arising at the junction with the beams. In a slab-and-girder floor it is usual to find beams running through the column in both directions. The column sections above the floor will in general be smaller than the section below so that any bars passing from one column to the next column above must be splayed inwards towards the centre-line. It is found most convenient to start the column bars immediately above a floor for reasons set forth in Chapter XII, which entails extending them about 20 diameters above the next floor higher. The percentages of steel and volume of links have already been dealt with in Chapter VI. The most convenient arrangement is a square column with four bars—one in each corner. The width of the column face should be at least 3 ins. more than the widest beam it carries, so that the corner bars will pass clear of all the beam bars. If more than four bars are used some of them will have to be passed up through, or to one side of, the beam bars. This point should be remembered when choosing a width for the beam section. The actual section may be calculated or taken from a list of tabulated values (see Chapter VI). If the column carries bending moment in addition to direct load the laps may want to be more than 20 diameters. This must be checked for each particular case. Some regulations call for a closer spacing of links near joints, the reason being apparently that a column whose ends are fixed in direction but not in horizontal position has most " buckling " moment at the ends. Links may be spaced evenly throughout and the author thinks that they should be continued through the beam intersections although they are often omitted to facilitate laying the beam bars.

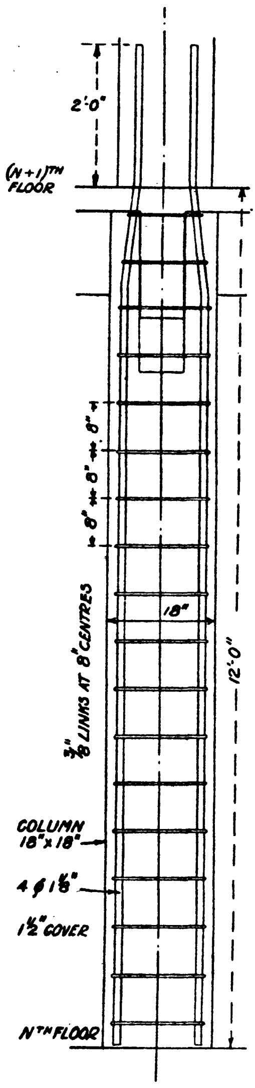


Fig. 150.

A typical interior column is shown in Fig. 150.

Exterior Columns.—In buildings the exterior columns are under somewhat worse conditions than the interior ones. The bending moment caused by the wind load is usually quite small, the wind pressure on the elevation being light. (Most of the wind load on structures is caused by discontinuity in the stream lines passing over the roof and round the corners. The wind diagrams that one sometimes sees showing uniform pressure on one slope of a roof and nothing on the other slope are merely diagrammatic and not intended to represent the true state of affairs.) The fact that the inside and outside faces may be at different temperatures will cause some bending stress in the column. In buildings it is usual to keep the outside face of the outside columns in one vertical line for the sake of appearances, which means that the centre line of the column changes as the section of column changes. Any eccentricity of loading which might ensue is modified by the floor. Against these three additional loads we have the fact that theoretically the outside columns are less heavily loaded by a continuous beam than by a series of simply supported beams.

If we calculate our column loads on the basis that the beams are simply supported, as is invariably done in practice, we shall theoretically have allowed for excessive load on the outside columns. Table CB VI shows that a distributed load on an end bay brings only ·433 *wl* on the end column against ·50 *wl* which we have allowed for. The author is in the habit of putting in a little more steel than calculated in the outside columns, say an additional 0·25%, although such a practice is by no means general. Where the columns show on the elevation of a building it looks better to keep a constant width all the way up.

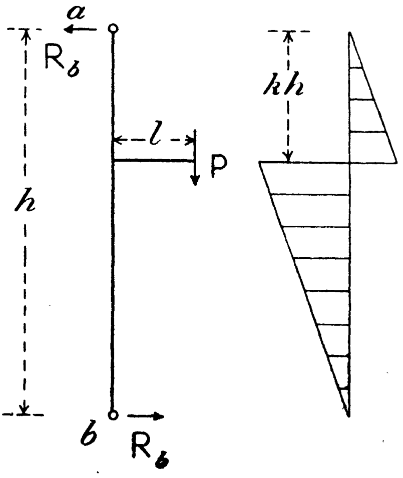


Fig. 151.

Brackets on Columns.—Occasionally a case occurs where a column carries a crane girder or other heavy load on a bracket. The main difficulty in such cases is to determine the conditions at the ends of the column. In a reinforced concrete structure they are neither fixed nor free but always some intermediate value which as a rule can only be estimated. In Fig. 151 *ab* is a column of height h hinged at top and bottom carrying a load P on a bracket of length l at a distance kh from the upper point. Since there is no horizontal load the horizontal reactions at top and bottom are equal and opposite and $= R_b$ say.

Moment in column just above the bracket $= R_b \times kh$.
,, ,, ,, ,, below ,, ,, $= R_b \times (1 - k)\, h$.
From the equilibrium of the whole column $R_b \times h = Pl$.

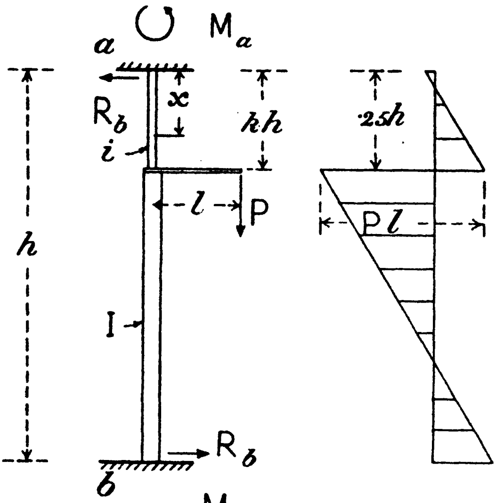


Fig. 152

Moment in column just above bracket $= k\,P\,l$
,, ,, ,, ,, below ,, $= (1 - k)\,P\,l$.

This is merely a problem in statics and is independent of the section of the column. If the column is fixed top and bottom we must know the moments of inertia as the solution depends on the relative deflections. In Fig. 152 let *ab* be a column of height h fixed top and bottom and let i be the moment of inertia above the bracket and let I be the moment of inertia below the bracket.

Let M_a and M_b represent the fixing moments. As before we see that the horizontal shear at top and bottom must be equal and opposite. Let this shear be equal to R_b.

For equilibrium of the whole system we have :—

$$P\,l + M_b = R_b h + M_a.$$

Working out the tangent slopes and sideways deflections by the slope-deflection method, we have three equations to solve the three unknowns M_a, M_b and R_b namely :—

$$P\,l + M_b = R_b h + M_a \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And by rearrangement :—

$$\frac{3\,M_b\,(1-k)^2}{I} - \frac{3\,M_a\,k^2}{i} = R_b\,h\left[\frac{k^3}{i} + \frac{(1-k)^3}{I}\right] \quad . \quad . \quad (2)$$

$$\frac{2\,M_b\,(1-k)}{I} + \frac{2\,M_a\,k}{i} = R_b\,h\left[\frac{(1-k)^2}{I} - \frac{k^2}{i}\right] \quad . \quad . \quad (3)$$

In any particular case the values k and $(1 - k)$ become fractions and easily obtained. The value $\frac{I}{i}$ is taken by trial and error and $P\,l$ is known.

For example if $k = \cdot 25$ and $I = 4\,i$ as might happen if the column carried little load from above, by putting in these numerical values and solving for M_a, M_b and R_b we get :—

(omitting the working)

$$\begin{aligned} M_b &= +\ \cdot 39\,P\,l \\ M_a &= -\ \cdot 0625\,P\,l \\ R_b\,h &= 1{\cdot}45\,P\,l. \end{aligned}$$

This gives the bending moment drawn in Fig. 152. Since the shear is constant the two parts of the bending moment diagram must be parallel, which furnishes a useful check on the figures.

Comparing the bending moments shown in Fig. 152 with the moments shown in Fig. 151, we see that the maximum moments resisted by the lower part of the column in these two cases are very nearly equal and the change produced by fixing the two ends of the column is slight. In Fig. 152 the smallness of the moment above the bracket is due to the small relative value of i which we have assumed as one-quarter of I. Only the relative values of i and I are required and bd^3 can be taken as a measure of the moment of inertia in both cases. In a practical case the author is in the habit of calculating the moments with free ends and with fixed ends and sketching them both on one diagram. After a cursory inspection of the joints top and bottom some intermediate line can be drawn in by eye, which may be taken as the actual bending moment.

The designer must make very sure that he does not confuse the case shown in Fig. 152, where the top of the column is fixed in both direction and position, with the case in Fig. 153 I, where the column head is fixed in direction only and may carry a lateral load (due to wind, say).

The case shown in Fig. 151 must also be distinguished from the case in Fig. 153 II. In both case I and case II in Fig. 153 the horizontal reactions top and bottom are equal to the applied load w and if no lateral load is applied the horizontal shear is constant throughout. Case I may be solved by using part of our previous analysis for we see that the slope at the bracket may be calculated from the top or from the bottom, as the slope at top and bottom is assumed to be zero.

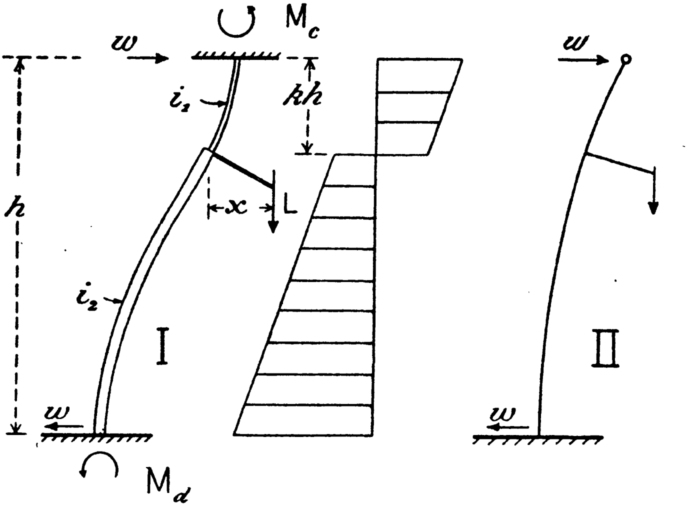


Fig. 153.

Allowing for the change in signs we have :—

$$M_c + M_d = wh + Lx$$

$$\frac{2M_c k}{i_1} - \frac{2M_d(1-k)}{i_2} = wh\left[\frac{k^2}{i_1} - \frac{(1-k)^2}{i_2}\right].$$

The solution of case II is sufficiently obvious. In all these cases, since the horizontal shear is constant, the slope of the bending moment diagram above the bracket is equal to the slope of the bending moment diagram below the bracket, which will check the working.

Should the column carry a lateral thrust at some point in its length the resulting moments may be established by comparison with a beam having similar end conditions and being similarly loaded. If the end conditions are somewhat vague, as almost invariably happens in practical cases, one or two variations should be sketched out and some intermediate value chosen.

Long Columns.—In the author's opinion it is desirable to base all calculations of working strength on the ultimate strength of the member. In the case of long columns it is essential to do so. The reason for this lies in the fact that the bending moment and shear forces are caused in part by the deflection of the column. As the load increases these deflections increase also and at an increasing rate. In a member which carries transverse load only, the actual value of the deflections, provided these are fairly small (as they invariably are in practice), has no effect on the bending moments or shears. Since the deflections increase at a very much faster rate when the working stresses are exceeded, it will be seen that the calculation of the deflections and stresses under working loads provides no criterion whatever of the ultimate strength of a column. The question, when compared with the design of structural steel columns (excluding solid round and solid square sections) is complicated by the fact that the elastic conditions of reinforced concrete vary greatly with change in stress intensity, whereas in steel the elastic conditions are almost constant up to the point of failure, which makes it fairly easy to calculate the ultimate strength of steel columns with simple end conditions and simple loads, both direct and transverse. We will commence by taking a simple case and then extend to more complicated loading. To simplify the matter still further we will start with a perfect elastic column and then attempt to apply the result to reinforced concrete.

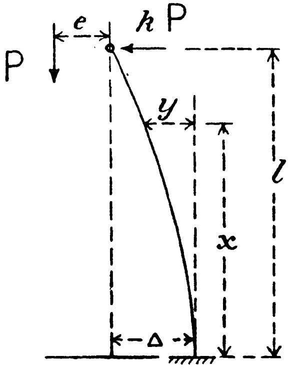


Fig. 154.

In Fig. 154 we have a column of height l which is fixed at the base so that the tangent at this point is always vertical. The top is free to move or rotate in any direction. The column carries a load P which is applied at a distance e away from the centre of the cap. There is also a transverse load at the cap which for convenience is written as kP, k being merely a numerical factor. The elevation in the figure shows the column bending about a principal axis the moment of inertia being constant and equal to I.

Take the foot of the column as origin and axes as shown. Let $\triangle$ = maximum deflection.

Bending moment at point $(x, y) = P(e + \triangle - y) + kP(l - x)$

$$EI\frac{d^2y}{dx^2} = P(e + \triangle + kl) - Py - kPx$$

$$\triangle = e(\sec ml - 1) + kl\left(\frac{\tan ml}{ml} - 1\right) \quad . \quad . \quad . \quad (1)$$

where $m = \sqrt{\frac{P}{EI}}$.

It will be seen that $\triangle$ tends to infinity when $ml = \frac{\pi}{2}$ giving Euler's Formula.

It is a notable fact and one that saves labour in designing long steel columns, that the ratio

$$\frac{\sec ml - 1}{\left(\frac{\tan ml}{ml} - 1\right)}$$

varies between 1·5 and 1·57 throughout its entire permissible range from $ml = 0$ to $ml = \frac{\pi}{2}$.

We may write therefore

$$\triangle = \left(e + \frac{2}{3}kl\right)(\sec ml - 1) \text{ within } 5\% \quad . \quad . \quad . \quad . \quad . \quad (2)$$

[Note.—If the value of $\frac{dy}{dx}$ at the base is not zero, that is to say, if the foundation yields under the load, then $\triangle$ in equation . . (1) must be increased by $i_0 l\left(\frac{\tan ml}{ml}\right)$ where i_0 is the slope at the base.]

We see at once that the value of $\triangle$ increases slowly for small values of m but much more rapidly for higher values.

We have now to determine the conditions of failure. All columns fail by excessive stress.

The idea that any column can be in a state of unstable equilibrium as laid down in Euler's Formula can be dismissed at once as utterly out of the question in all practical cases. No practical column is quite straight, quite uniform or quite centrally loaded. Any load therefore, however small, causes the column to deflect. As the load increases this deflection increases and at an increasing rate until the stress at some point exceeds the ultimate sectional stress and local failure by tension or by crushing-out develops.

The ultimate sectional stress is not necessarily the ultimate stress as shown by a short test-piece in a testing machine. (In steel columns made up of thin plates or sections the ultimate sectional stress is equal to the compressive elastic limit or U.L.P.) In reinforced concrete columns, tests seem to show that short pieces fail when the concrete reaches its ultimate strength of, say, 2400 lbs. per sq. in. while the rods are at the elastic limit say 16—20 tons per sq. in., slipping of the rods in the concrete having taken place. Failure by tension is not so likely to occur. In fact as we have seen in cases of combined compression and bending a percentage of about 1·0% top and bottom, or 2·0% in all, in a rectangular section, will prevent the tensile stress under working load exceeding $\frac{18,000}{750}$ times the concrete stress even if all the applied stress is transverse and the direct load is zero. We shall therefore for the moment ignore failure by tension. Our condition of failure is, therefore, that the extreme fibre stress at the most highly stressed section does not exceed 2400 lbs. per sq. in. We have yet to determine our value of $E\,I$. In an ordinary rectangular section under conditions of loading such as shown in Fig. 154 this would vary from section to section in accordance with the variation of stress.

The sections near the top of the column have much less bending stress than those near the bottom. It follows therefore from Table C III (a) that the neutral axis depth grows less as we approach the foot of the column and consequently I decreases. Moreover the value of E decreases as the stress increases. In the case we have chosen in Fig. 154 the most important sections, as far as deflection goes, are those near the foot of the column since any change of slope at the base causes a large movement of the cap. A change of slope near the top would cause little movement of the cap. In our analysis we have assumed that $E\,I$ is constant, and although we cannot realize this condition in practice we can by choosing some intermediate value of $E\,I$ arrive at a value for $\triangle$. As no definite information on the subject is obtainable the author proposes to use a value of $1,500,000 \times \frac{ba^3}{12}$ where b = breadth and a = overall depth of the section for $p = 2\%$ (see further discussion). We have still to determine a value for e, the eccentricity. All practical columns vary in texture due to inequalities in mixing and placing, yielding of centering, etc. This means that the centre of actual resistance will not coincide with the geometrical centre of the section as drawn.

Thus we have always an accidental eccentricity in much the same way as with steel columns. The actual value of this eccentricity will vary from case to case and its determination is always obscured by other effects. The author has assumed a value of $\frac{a}{15}$ as from Table C II we see that for $p = 2\%$ and $\frac{e}{a} = {\cdot}066$ the working load for a value of $f_c = 600$ is 575 ab which corresponds to $\frac{600}{800}$ of 768 (the working stress when $e = 0$). This value is taken as experience shows that a value of $f_c = 800$ lbs. per sq. in. may be worked to on eccentrically loaded short columns, while only 600 lbs. per sq. in. may be taken on " centrally " loaded short columns. Having settled the values of all the unknowns we have only to find a method of applying these results. We must first of all decide on the value of the factor of safety. This we shall assume as 3 for a standard value. Other values can be obtained fairly readily from our standard results. We have :—

(Maximum Stress under 3 times working load applied at the ultimate deflection) = 2400 lbs. per sq. in.

Divide both sides of this equation by three.

(Maximum Stress under working load applied at the ultimate deflection) = 800 lbs. per sq. in.

This demonstrates fairly clearly that we must take the ultimate deflection, i.e., the deflection under three times the working load. This is made clearer from a consideration of the two cases shown in Fig. 155.

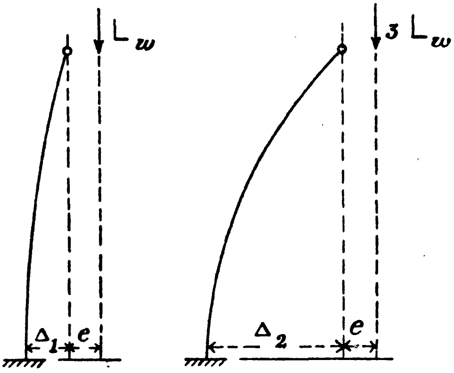


Fig. 155.

The left-hand view shows a column under its working load L_w while the right-hand view shows the same column under its ultimate load. If the column is very stiff $\triangle_2$ is at least three times $\triangle_1$ and if the column is not very stiff (which we have assumed as a basis for this argument) $\triangle_2$ may be 4, 5, 10 times, etc., as big as $\triangle_1$.

Suppose the column is 10 ins. × 15 ins. in section with 2% of steel and suppose $\triangle_1 = \frac{1}{4}$ in. while $\triangle_2 = 2{\cdot}5$ ins. By assumption $e = 1$ in. Suppose $L_w = 109{,}000$ lbs.

Under working load at the base section the total eccentricity = 1·25 in.

In Table C II, $\frac{e}{a} = \frac{1{\cdot}25}{15} = {\cdot}083.$

$$p = 2\%,\ \frac{L}{ab} \times \frac{600}{f_c} = 540, \quad \therefore\ f_c = \frac{109{,}000}{10 \times 15} \times \frac{600}{540}$$

= 809 lbs. per sq. in. which is safe.

Under ultimate load we have a total eccentricity of 3·5 ins.

In Table C III, $\frac{e}{a} = \frac{3{\cdot}5}{15} = {\cdot}233.$

$$p = 2\%, \ \frac{L}{ab} \times \frac{600}{f_c} = 345 \quad \therefore f_c = \frac{3 \times 109{,}000}{10 \times 15} \times \frac{600}{345}$$

= a calculated value of 3800 against an allowable value of 2400 lbs. per sq. in.

In this example, even if we had assumed the column to be extremely stiff so that $\triangle_2 = 3\ \triangle_1 = \frac{3}{4}$ in. we should get a calculated value for the ultimate stress of 2700 lbs. per sq. in.

Special Case.—" Central " Loads. Rectangular Sections.—If we have nothing but a direct load to carry we can determine the value of I much more accurately than we can in the general case of combined direct and transverse loads. It must be understood that we only require I as a measure of the radius of curvature at the point in order to calculate the deflection. All that we want is the ratio of increase of strain in proportion to the depth of the section. If we know the values of f_c, E_c and na we can at once determine our value $E \times \frac{d^2y}{dx^2}$.

If we also know the value of the bending-moment we can, by dividing one expression by the other obtain a value for I. If we write

$$I = v \times \frac{ba^3}{12}$$

we can readily calculate v for our special case if we know the amount of steel and its distribution.

If no part of the section is cracked we have in Fig. 156 if p is

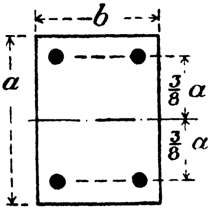


Fig. 156.

the total percentage and $m = 15$, by first principles.

$$\text{Total } I \text{ in concrete units} = \frac{ba^3}{12} + \frac{p\,a\,b}{200} \times \left(\frac{3}{8}a\right)^2 \times 14 \times 2$$

$$= \frac{ba^3}{12}\left[1 + \frac{p}{4{\cdot}22}\right] \text{which if } p = 2\%, = \frac{ba^3}{12} \times 1{\cdot}47.$$

If the section is cracked we obtain I from the values used for Tables C III. We have :—

$$\text{if} \quad n = {\cdot}60$$
$$f_c = 600$$
$$p = 2\%.$$

Then $$\frac{L}{ab} = 205 \text{ and } \frac{e}{a} = \cdot45.$$

Now $$\frac{E}{\rho} = \frac{f_c}{n\,a} \text{ by first principles}$$

and $$E\,I\frac{d^2y}{dx^2} = M \quad \text{where } \frac{d^2y}{dx^2} = \frac{1}{\rho}$$

$$E\,I\frac{d^2y}{dx^2} = I \times \frac{f_c}{n\,a} = M.$$

Now $\frac{f_c}{na} = \frac{600}{\cdot60\,a}$ and $M = 205\,ab \times \cdot45\,a$

$$\therefore\ I = \frac{M \times n\,a}{f_c} = \frac{205\,ab \times \cdot45\,a}{600} \times \cdot60\,a = ba^3 \times \cdot092$$

$$= \frac{ba^3}{12} \times 1{\cdot}10 \text{ against } \frac{ba^3}{12} \times 1{\cdot}47 \text{ for the uncracked section.}$$

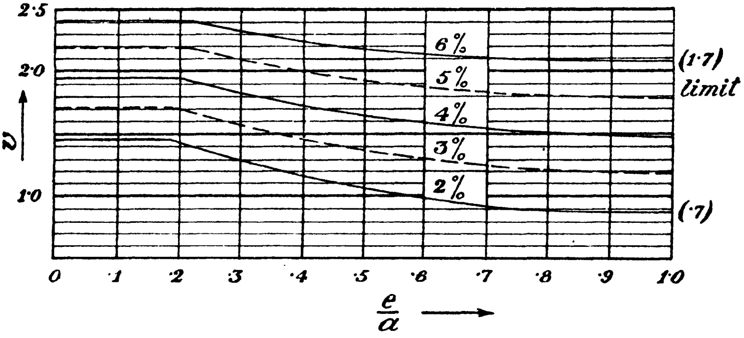


Fig. 157.

The curve in Fig. 157 shows the value of I for different values of p and n. (The reader should construct a similar table for himself from Table C III a where values of n are given on the table.)

Our general analysis gives us

$$\Delta = e\,(\sec m\,l - 1).$$

Total eccentricity at critical point

$$= e + \Delta = e\,(\sec m\,l - 1) + e = e \sec m\,l.$$

If we write :—

$$\text{accidental eccentricity} = \frac{a}{15}$$

$$(e + \Delta) = \frac{a}{15} \sec ml = \frac{a}{15} \sec\left(l\sqrt{\frac{P}{EI}}\right) \text{ (in radians).}$$

Using a factor of safety of three and writing

L = working load

$P = 3\,L$ (see Fig. 154).

$$I = v \times \frac{ba^3}{12} \text{ and } E = 1{,}500{,}000 \text{ lbs. per sq. in.}$$

we have :—

$$(e + \triangle) = \sec\left(\cdot 28\,\frac{l}{a}\sqrt{\frac{L}{vab}}\right)^{o} \times \frac{a}{15} \text{ (in degrees).}$$

If we choose a value for $(e + \triangle)$ in terms of a, and values for p and f_c (which we have already settled on as 800 lbs. per sq. in.) we can obtain v as above, getting thence the value of the angle. (Having settled $(e + \triangle)$ and f_c, Table C III gives us the value of $\frac{L}{ab}$.)

For example if $p = 2\%$ and $n = \cdot 60$

with $f_c = 800$ we get $\frac{L}{ab} = 273$ and $(e + \triangle) = \cdot 45\,a$

$$v = 1\cdot 1$$

$$\therefore \quad \cdot 45\,a = \sec\left(\cdot 28\frac{l}{a}\sqrt{\frac{273}{1\cdot 1}}\right)^{o} \times \frac{a}{15}$$

$$\sec\left(\cdot 28\frac{l}{a}\sqrt{247}\right)^{o} = 6\cdot 75$$

$$\therefore \quad \cdot 28\,\frac{l}{a} \times 15\cdot 7 = 81^\circ\,30' \text{ giving } \frac{l}{a} = 18\cdot 5$$

Referring to Fig. 154 we see that we have taken our value for I at the most highly stressed section and are therefore on the safe side. Also the value $\frac{l}{a}$ is only half the equivalent round-ended length, which $= 2 \times 18\cdot 5 = 37\cdot 0$.

Therefore a square column or a rectangular section free to bend about one axis having 2% of reinforcement and having an equivalent round-ended length of 37·0 × breadth (measured in the direction of freedom) will safely support a load of 273 lbs. per sq. in. calculated on the area ba, provided the steel is distributed as on Table C I which shows the safe working loads on long columns equivalent to a stress of 600 lbs. per sq. in. on short columns. (This stress will really only give a factor of safety of three on a practical column under practical conditions of loading and not a factor of four as might be supposed from Chapter VI.)

Where a column has the same unsupported length on both axes a square or circular section naturally suggests itself. The ratio $\frac{l}{a}$ in Table C I may be written as $\frac{1\cdot 2\,l}{D}$ in the case of a circular column. Where a column is braced frequently in one direction but free in the other, the least width should be made about one-twentieth of the height between the bracings centre to centre. Suppose a column is 60 ft. 0 ins. high and braced in one direction only, every 20 ft. 0 ins. of its height. Then it may be made about 12 ins. wide.

Its depth must be found from Table C I taking l as some fraction of 60 ft. 0 ins. (depending on the end conditions) and a as the depth.

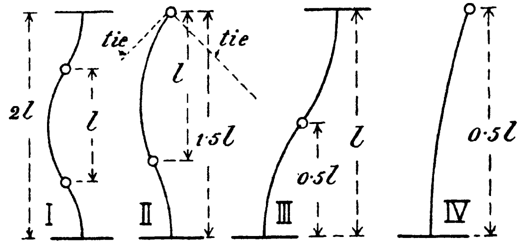
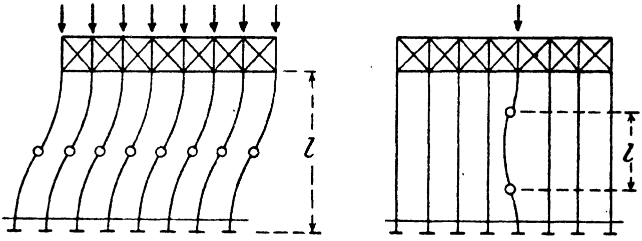


Fig. 158.

End Conditions.—Round-ended Length.—The exact determination of the end conditions is the most difficult part of all column design. No practical column is immovably fixed at its ends although a column well bonded to a large heavy base is very close to the ideal conditions. On the other hand no column is actually free at any joint with other members. The actual end conditions vary between fixed, flat and continuous. A few cases are illustrated in Fig. 158 which shows the equivalent round-ended length l in each case. Great care must be taken to distinguish between cases I and II where the head of the column is fixed in position and cases III and IV where the column head may move laterally as it is easy to confuse case I with case III. Referring to Fig. 159 which shows the same row of columns under different loading. It is clear in case (*a*) where all the columns are loaded that they may all deflect in one direction giving an equivalent round-ended length equal to their height. In case (*b*) where only one column is loaded the stiffness of the unloaded columns will prevent the top part of the structure moving bodily



(*a*) Fig. 159 (*b*)

and the equivalent round-ended length of the loaded column is about half the height.

Table C I.—The working loads in this table are given for round-ended lengths up to 60 diameters. Some authorities limit the length to 40 diameters as an absolute maximum although some pile jetties certainly exceed this value. (In the case where the piles are driven through a short depth of silt on to hard rock and braced two or three feet above L.W.O.S.T. their equivalent round-ended length when the jetty is fully loaded is twice the height from shoe to bracing. Using 14 ins. × 14 ins. piles a height of 35 ft. 0 ins. would be nothing out of the common and this is equivalent to 60 diameters round-ended length.) Before using this table the designer must make very sure that the column in question is absolutely free from lateral load. If the end conditions are rather vague two or three alternatives should be tried assuming different conditions each time. If any doubt is felt about a final selection take the largest. The case of long columns occurs so rarely (except in pile work) that there is no standard practice. The author's experience with long piles confirms the values in Table C I. It is obvious that when the problem does arise the practical designer cannot evade the issue and must decide on some definite section.

Long Columns under Eccentric Loads.—One Free Axis.

When the column carries a series of lateral loads these may be replaced by a single lateral load by taking moments about the fixed end. In Fig. 160 the loads P_1, P_2, etc., may be replaced by a single load P at the cap where

$$P\,l = P_1 h_1 + P_2 h_2 + \;.\;.\;.\;.\;.$$

This is on the safe side. Assuming a factor of safety of three the general principles of design are very simple. All we have to do is to work out the deflections caused by a direct load 3 L and a lateral load 3 P acting together which will give us the eccentricities at all sections. The sections can be checked by Tables C II and C III for a load L applied at our calculated eccentricities allowing $f_c = 800$. In working out the eccentricities we must add one fifteenth of the

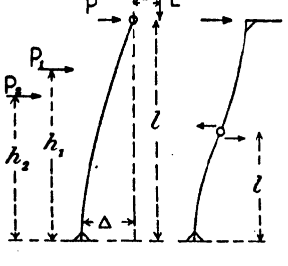


Fig. 160.

diameter to the applied eccentricity at the cap to cover accidental eccentricity as explained for " central " loads.

The total bending moment at the base is

$$L \times (e + \Delta) + Pl$$

where $$\Delta = \left(e + \frac{2}{3}\,\frac{P}{L}\cdot l\right)\left(\text{sec.}\left({\cdot}28\,\frac{l}{a}\sqrt{\frac{L}{vab}}\right)^{\circ} - 1\right).$$

The value of v can be found from Fig. 157 and should be taken at the most highly stressed section.

For a preliminary estimate we may write $v = 1$ for $p = 2\%$ and $v = 2$ for $p = 6\%$. (The values of v in Fig. 157 agree with the values assumed on Tables C II and C III. The value of l is the value shown on Fig. 160 and is only half the round-ended height. All units are in pounds and inches.)

Example.—A column is 20 ins. × 10 ins. section with four bars 1⅛ in. diameter. It is 30 ft. 0 ins. high and is efficiently braced on its weak axis to neighbouring columns at frequent intervals. It has no bracing of any kind on its strong axis, the cap being free to turn and move in that direction.

It carries a load of 5 tons applied to the cap at a distance of 15 ins. from the centre line. It also carries a load of 1300 lbs. evenly distributed along its height tending to bend it in the same direction as the eccentricity.

Since the column is braced efficiently on its weak axis we need not trouble about bending in that direction.

We have

$$a = 20 \text{ ins.}$$

$$b = 10 \text{ ins.}$$

$$p = \frac{3{\cdot}98}{200} \times 100 = 2\%.$$

From Fig. 157 put $v = {\cdot}8$.

Replace the distributed load of 1300 lbs. by a load of 650 lbs. at the cap

$$L = 11{,}200 \text{ lbs. and } k = \frac{650}{11{,}200} = {\cdot}058 \left(\text{where } k = \frac{P}{L}\right).$$

Deflection at foot $= \Delta$

$$= \left(e + \frac{2}{3} \times {\cdot}058 \times 360''\right)\left(\sec\left({\cdot}28 \times \frac{360}{20}\sqrt{\frac{11{,}200}{10 \times 20 \times {\cdot}8}}\right)^{\circ} - 1\right)$$

$$e = 15'' \text{ applied} + \frac{20}{15} \text{ accidental} = 16{\cdot}3 \text{ ins.}$$

$$\Delta = (16{\cdot}3 + 14)\,(\sec\,({\cdot}28 \times 18 \times 8{\cdot}3)^{\circ} - 1)$$

$$= 30{\cdot}3 \times (\sec 41{\cdot}7^{\circ} - 1) = 30{\cdot}3 \times {\cdot}33 = 10 \text{ ins.}$$

Total bending moment at foot of column

$$= L \times (e + \Delta) + P \times l \text{ (as in Fig. 160)}$$

$$= 11{,}200 \times (16{\cdot}3 + 10) + 650 \times 360$$
$$= 294{,}000 + 234{,}000$$
$$= 528{,}000 \text{ lbs.-ins.}$$

Referring to Table C III (*a*) we have at the base section

$$\frac{L}{ab} \times \frac{600}{f_c} = \frac{11{,}200 \times 600}{10 \times 20 \times f_c}.$$

$$\text{Total eccentricity} = \frac{528{,}000}{11{,}200} = 47 \text{ ins.}$$

$$\frac{e}{a} \text{ (on table)} = \frac{47}{20} = 2{\cdot}35.$$

For $p = 2\%$ the table gives

$$\frac{L}{ab} \times \frac{600}{f_c} = 49 = \frac{11{,}200 \times 600}{10 \times 20 \times f_c}$$

$$\therefore\ f_c = \frac{11{,}200}{200} \times \frac{600}{49} = 685.$$

If this column had been centrally loaded only, as the ratio

$$\frac{l}{a} = \frac{360}{20} \times 2 = 36$$

it would carry 280 × 10 × 20 = 56,000 lbs. (Table C I).

If we had designed the column as a vertical cantilever only, to a stress of 685 we should have :—

applied bending moment at base $= 650 \times 360 + 11{,}200 \times 15$

$$= 234{,}000 + 168{,}000 = 402{,}000$$

$$R = \frac{402{,}000}{10 \times 17{\cdot}5^2} = 131$$

$$\frac{R}{f_c} = 0{\cdot}191$$

From Table RB IV, a value of 0·7% top and bottom would cover this, against 1·0% as supplied in the example.

Long Columns under Eccentric Loads.—Bending about Both Axes.—The accidental eccentricity may be taken as one fifteenth the smaller diameter bending about the weaker axis. The deflections about the minor and major axes may be worked out independently with sufficient accuracy, but the checking of the final section, unless a square or round section is adopted, requires a lot of trial and error as we may have a rectangular section with the load applied near one corner. In very complicated cases the deflections may be taken off in a manner similar to those for a cambered strut. In such cases where we are working more or less by guesswork an ample factor of safety should be allowed. Such complicated cases occur very rarely as the designer will do his best to avoid any arrangement of members that might cause them. They vary so much, one from another, that each must be treated on its merits. Sometimes the whole thing may be simplified by splitting a larger section into two smaller ones braced together.

Limiting Deflection of a Straight-Cambered Strut.—With Transverse Loads.—In cases where the end loads are small and the transverse loads are large the bending moment will vary greatly and the continuation of the maximum section throughout the whole length of the strut is wasteful. In the ideal case we could vary the section so that at every point our maximum allowable values of f_c and f_s were realized. This would result in a simple relation between the effective depth and the radius of curvature.

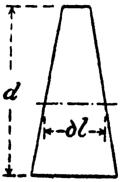


Fig. 161.

In Fig. 161 suppose we have a short length of beam $\delta\ l$ having a stress f_c at its upper edge and a stress f_s at its lower edge.

Total strain at top edge $= \dfrac{f_c}{E_c}\ \delta l$

Total strain at bottom edge $= \dfrac{f_s}{E_s}\ \delta\ l$

Rate of change of strain $= \left(\dfrac{f_c}{E_c} + \dfrac{f_s}{E_s}\right)\delta\ l \times \dfrac{1}{d} = \dfrac{\delta l}{\rho}$

$$\therefore\ \frac{1}{\rho} = \frac{d^2y}{dx^2} = \left(\frac{f_c}{E_c} + \frac{f_s}{E_s}\right) \times \frac{1}{d}.$$

If we vary our section such that we obtain constant values of f_c and f_s the radius of curvature varies as the depth d.

Let Fig. 162 represent such a strut whose depth at the centre is a and at each end is b the half length being l.

Take the centre point as origin.

At any point x the depth is given by $a - \dfrac{(a-b)\,x}{l}$ the profile being a straight-line.

We have already $\dfrac{d^2y}{dx^2} = \left(\dfrac{f_c}{E_c} + \dfrac{f_s}{E_s}\right) \times \dfrac{1}{(\text{depth})}.$

Write this as $\dfrac{d^2y}{dx^2} = \dfrac{k}{(\text{depth})}$ where k is a constant.

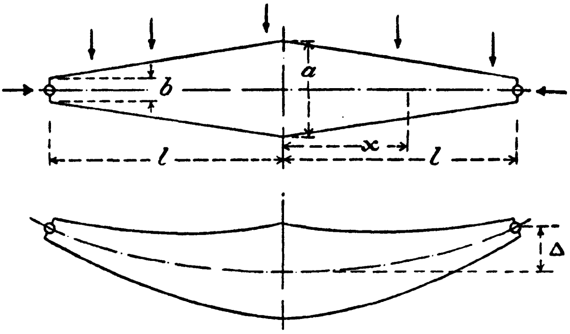


Fig. 162.

Then $$\frac{d^2y}{dx^2} = \frac{k}{d} = \frac{k \times l}{al - (a - b)\, x}.$$

Finally if we write $b = z\,a$ where z is a fraction

$\Delta = \frac{kl^2}{a\,(1 - z)^2}\left[(1 - z) + z \log z\right]$ the logarithm being hyperbolic.

If $z = \frac{b}{a} =$	1·0	·9	·8	·7	·6	·5	·4	·3	·2	·1
$\Delta = \frac{kl^2}{a} \times$	·50	·53	·54	·56	·58	·61	·645	·69	·75	·825

If z were $= 0$ which is hardly possible the factor would be 1·00.

We have now to determine the value of k we are going to use. We may assume that $f_c = 2400$ and $E_c = 1{,}500{,}000$. It seems to the author that in a member where the steel stress is maintained throughout the whole length, that failure by tension would occur sooner than in a member where the maximum stress occurs over a short length only. This would be particularly the case where the top boom and bottom boom were separate members joined together at intervals only. In such a case if the steel in the bottom boom passed the elastic limit the whole boom would be in a state of disintegration and all adhesion and anchorage would vanish.

The author therefore proposes to use $f_s = 40{,}000$ and $E_s = 30{,}000{,}000$.

This makes
$$k = \left(\frac{2400}{1{,}500{,}000} + \frac{40{,}000}{30{,}000{,}000}\right) = \frac{48 + 40}{30{,}000} = \frac{2{\cdot}93}{1000}.$$

It will be seen that this calculated deflection is the absolute maximum that could be caused by any arrangement of loading as the radius of curvature at all points is a minimum. In a practical strut of this outline the bottom boom and top boom would most probably have the same concrete area throughout and the maximum stresses would only be realized at one or two sections. We shall therefore always be on the safe side by taking this value for the deflection (ultimate).

Example.—A strut lies in a horizontal plane and is 120 ft. 0 ins. long. The ends may be taken as round. The strut being one of a pair is braced efficiently to its fellow so that neither can deflect horizontally. The estimated self-weight including bracing, etc., is 2000 lbs. per foot which may be taken as uniformly distributed. The strut carries a central end thrust of 50 tons. Design it for a factor of safety of three.

We shall use a strut as shown in Fig. 163 and as we assume that the stress in each boom section is uniform we shall take the height centre to centre of booms. We will try a height of 6 ft. 6 ins. in the centre of the span and 2 ft. 6 ins. at the ends

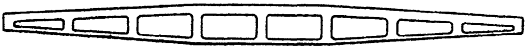


Fig. 163.

$$a = 6{\cdot}5', \quad b = 2{\cdot}5', \quad \frac{b}{a} = {\cdot}385, \quad l = 60'\,0''$$

$$\Delta = \frac{k\,l^2}{a} \times {\cdot}66 = \frac{2{\cdot}93}{1000} \times \frac{120 \times 120}{4 \times 6{\cdot}5} \times {\cdot}66 = 1{\cdot}06 \text{ feet.}$$

Bending moment at centre due to transverse load

$$= 2000 \times \frac{120^2}{8} = 3{,}600{,}000 \text{ lbs.-ft.}$$

due to end thrust $= 50 \times 2240 \times 1{\cdot}06 = 117{,}000$ lbs.-ft.

Thrust on centre line at mid span $= 112{,}000$ lbs.

Total compression in top boom

$= 56{,}000$ lbs. direct $+ 570{,}000$ bending $= 626{,}000$ lbs.

Total tension in bottom boom

$= -\,56{,}000 + 570{,}000 = 514{,}000$ lbs.

$$A_s \text{ in bottom boom} = \frac{514{,}000}{40{,}000} \times 3 = 38{\cdot}5 \text{ sq. ins.}$$

Taking $p = 2\%$ for the top boom and working to 768 lbs. per sq. in. to cover local accidental eccentricity (see Table C I).

$$\text{Area of top boom} = \frac{626{,}000}{768} = 816 \text{ sq. ins.}$$

Say 24 ins. deep by 34 ins. wide.

Area of steel $= 16{\cdot}3$ sq. ins.

Bottom boom say 12 ins. deep × 34 ins. wide.

Most cambered sections have a parallel piece in the middle as shown so that our calculated deflection is high. The additional stress due to the deflection in this case is small but it should always be allowed for. We have examined the mid-span section only but others must be examined and the boom sections reduced where possible. The curve of deflections may be taken as a parabola.

The cases we have treated so far have been cambered in one elevation only, nothing being said about deflections in a plane at right angles. If the reader should ever come across a case he must do the best he can from first principles. A fairly frequent application of cambered struts occurs in telegraph poles, etc.

Long Columns. Conclusion.—The author admits that his treatment is somewhat illogical as he uses the ultimate deflection and then designs the section for a straight-line law of stress assuming $m = 15$ after having assumed $E_c = 1{,}500{,}000$. This was rendered necessary by the fact that English practice always uses a straight-line law and always calculates the stresses under working loads. Realizing that it is not possible to alter the methods of design in general use except by a slow process of gradual evolution he has tried to make his methods fall into line with standard practice. It is impossible to discuss all the cases that may occur and considering the rare occurrence of long columns quite sufficient space has been devoted to them.

As far as the author is aware this method of calculating the strength from the limiting ultimate deflection (the method is known as the L.U.D. method) was and still is the only method in existence of designing long struts under practical loading. This is the reason why it has been set out in detail. The figured results are merely intended to convey to the reader the order of magnitude of the various effects. On such a subject and with such a variable material as concrete any pretence at accuracy is out of the question.

Quantities in Columns.—In taking off preliminary quantities the heights are taken gross where columns pass through floors. In a building the column heights are taken from top of floor to top of floor, no deduction being made for the thickness of floor slab or intersections with beams.

The quantity of concrete per yard of column is the sectional area in square inches divided by 36^2, no allowance being made for the volume occupied by the steel. The centering per yard of column is the perimeter in inches divided by 36. Centering for circular columns will be separated from that for square columns.

The steel may be estimated separately if time permits. The main bars are lapped about 20 to 25 diameters so that long thin columns will have less percentage lap than short stout ones. With 15% lap we have a factor of $1{\cdot}15 \times 3 \times 3{\cdot}4 = 11{\cdot}8$ for main bars only.

The links are about 0·2% for 1% of main steel and will therefore be one-fifth to one-sixth of the main bars. For normal design a combined steel factor of 15 is sufficient. Thus if a column has four bars 1 in. (3·14 sq. ins.) the weight of steel per yard of column including laps and links is about 15 × 3·14 or 47 lbs.

TABLE C I (*continued*)

NUMERICAL REDUCTION FACTORS FOR LONG COLUMNS.

(Total main steel not less than 2%).

For columns up to 15 diameters no reduction need be allowed.

$\frac{l}{a}$	Reduction Factor	$\frac{l}{a}$	Reduction Factor	$\frac{l}{a}$	Reduction Factor
16	0·89	32	0·51	48	0·18
18	0·85	34	0·43	50	0·17
20	0·82	36	0·37	52	0·15
22	0·78	38	0·32	54	0·14
24	0·73	40	0·29	56	0·13
26	0·69	42	0·25	58	0·12
28	0·63	44	0·22	60	0·11
30	0·58	46	0·20		

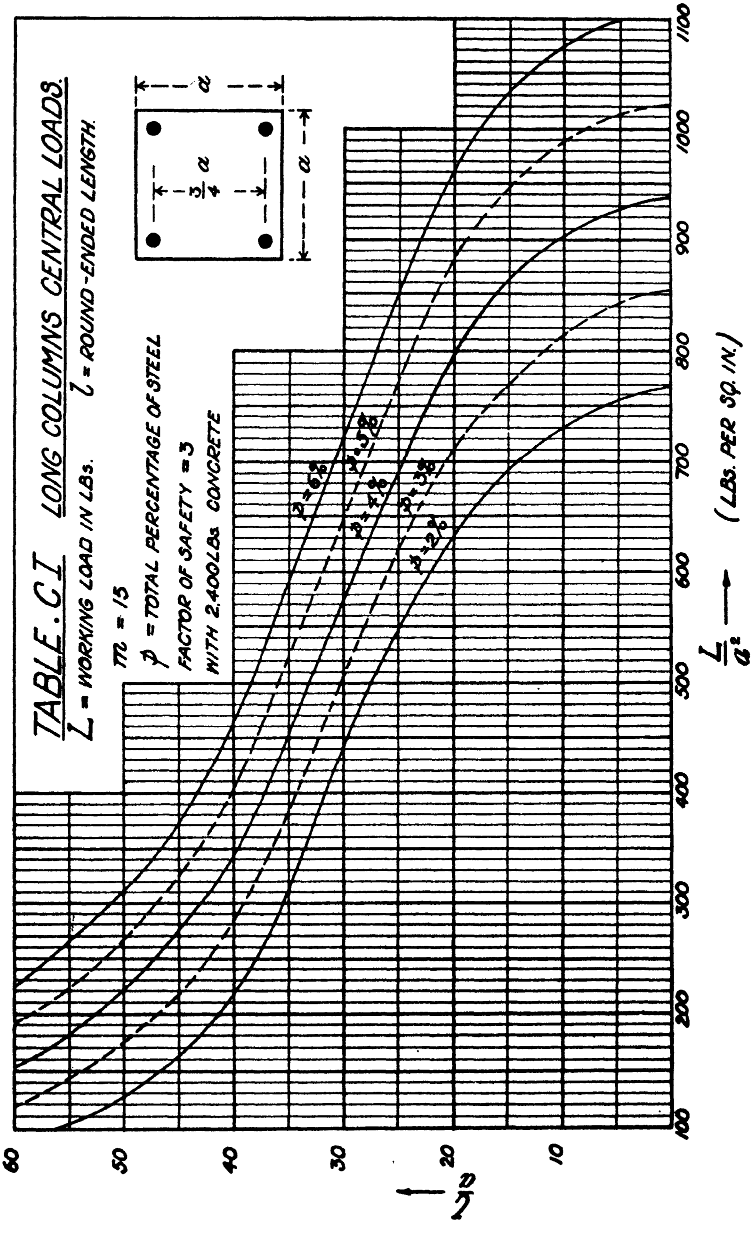


TABLE. CI
LONG COLUMNS CENTRAL LOADS.
L = WORKING LOAD IN LBs.
l = ROUND-ENDED LENGTH.
m = 15
p = TOTAL PERCENTAGE OF STEEL
FACTOR OF SAFETY = 3
WITH 2,400 LBs. CONCRETE
a
a
3/4 a
p = 6%
p = 5%
p = 4%
p = 3%
p = 2%
100
200
300
400
500
600
700
800
900
1000
1100
L/a² →
(LBs. PER SQ. IN.)
60
50
40
30
20
10
l/a ←

CHAPTER XVII

MESH-PANEL SLABS

Theory of Square Isolated Panel under Uniform Superload.— It is possible, although beyond the scope of practising engineers, to calculate the moments and shears in a "medium thick" slab of perfectly elastic material resting on ideal theoretical supports under certain simple cases of loading.* Even if we could easily solve the general case of such an ideal slab under partial or point loads, the result would have little bearing on the problem in question, namely the practical strength of a reinforced concrete slab supported on reinforced concrete beams. As in the case of flat-slab floors our only guidance are those average or total moments that can be calculated in simple cases. Many of the earlier writers on the subject fell into the error of regarding a slab supported on four edges as two separate slabs each supported on two edges. There are several different theories of this type, all at variance with one another, but all have the same fundamental weaknesses—insufficient attention to the maximum moments (which occur across the diagonals and *not* parallel to the sides) and much heavier steel in one direction in rectangular panels than the other despite the fact that the theories are based on the assumption of a slab of absolutely uniform strength in all directions.

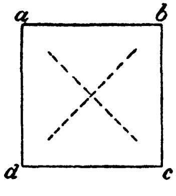


Fig. 164

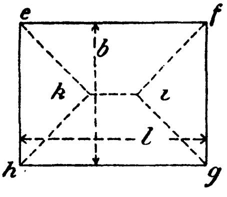


Fig. 165

If we have a uniform slab *a b c d* in Fig. 164 supported on four beams *ab cd bc* and *da* all of the same span and same stiffness and if this slab carries a uniform load *w* over its whole area we may readily show that the total moment on section *ac* or *bd* is $\frac{\sqrt{2}\, w\, l^2}{24}$ or the *average* moment per unit length of *ac* or of *bd* is

$$\frac{wl^2}{24} \text{ or } \frac{W}{24} \text{ where } W = wl^2.$$

---

* See Kelvin and Tait *Natural Philosophy*.

This simple theorem may be regarded as all that we really know about the strength of reinforced concrete mesh panels. It will be seen that we only know the *average* unit moment but by the Principle of Greatest Strength we may reinforce the slab all over for this average moment, and assume that at failure the moments will average themselves out. (This idea is supported by loading tests on mesh-panel slabs.)

Rectangular Isolated Panel.—On account of the great uncertainty caused by the unequal deflection of the supporting beams, mesh panels should not be used if the length exceeds 1·5 times the breadth. They are not economical in any case (if we include the beams in both directions) beyond this ratio.

If Fig. 165 represents a rectangular panel, uniformly loaded, we may assume (and loading tests confirm) that at failure the slab breaks along lines $e\,k$, $k\,h$, $k\,i$, $i\,f$ and $i\,g$ and that the unit moment across these lines is constant. This gives a unit moment of very nearly $\frac{w\,b\,l}{24}$ or $\frac{W}{24}$ where $W = w\,b\,l$.

We may say, therefore, that the maximum unit moment in an isolated mesh panel uniformly loaded whether square or rectangular is $\frac{W}{24}$ where W is the total load on the whole panel.

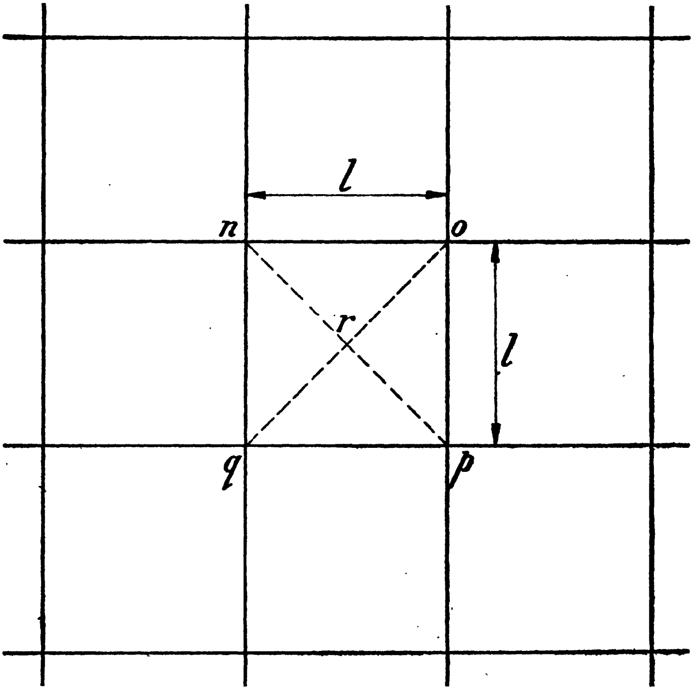


Fig. 166

Continuous Panels under Uniform Superload.—Let Fig. 166 represent a square panel $n\ o\ p\ q$ surrounded on all sides by exactly similar panels. Firstly assume that every panel of the floor is covered by a uniform superload w. There will be no shear forces on sections np, oq. On section nq there will be a shear force $\frac{W}{4}$ where W is the total load on one panel. Take the equilibrium of the quarter panel nrq. It has :—

A vertical shear $\frac{W}{4}$ on nq

A load $\frac{W}{4}$ distant $\frac{l}{6}$ from nq

Moments on sections nq, nr and rq.

If M_1 is the average unit negative moment on section nq and M_2 is the average unit positive moment on nr and rq (taking moments about nq)

$$M_1 \times nq + \frac{M_2\,(nr + rq)}{\sqrt{2}} = \frac{W}{4} \times \frac{l}{6}$$

$$M_1\,l + M_2\,l = \frac{Wl}{24}.$$

$$M_1 + M_2 = \frac{W}{24}.$$

Assuming that with all panels loaded $M_1 = 2M_2$

then $$M_1 = \frac{W}{36}.$$

Similarly if we load panel $nopq$ alone and remove all load from the adjoining panels we still have

$$M_1 + M_2 = \frac{W}{24}.$$

Assuming that with this distribution of load $M_2 = 2M_1$

$$M_2 = \frac{W}{36}.$$

These are merely crude average values but reinforced concrete is still a crude material, and the thickness of the slab, the strength of the concrete and its elastic properties will vary greatly from panel to panel even under working loads. At breaking load all resemblance to the theoretical, elastic, " medium thick " slab vanishes, the moments average out and only those values which are statically determinate persist. For all building work the author has for many years used the coefficients given in Fig. 167 and has no hesitation in recommending them for general use. As they are, to some extent, based on the Principle of Greatest Strength it is as well to increase the negative moments by 20% for watertight or gas-tight work to cover unequal distribution of moment under working loads.

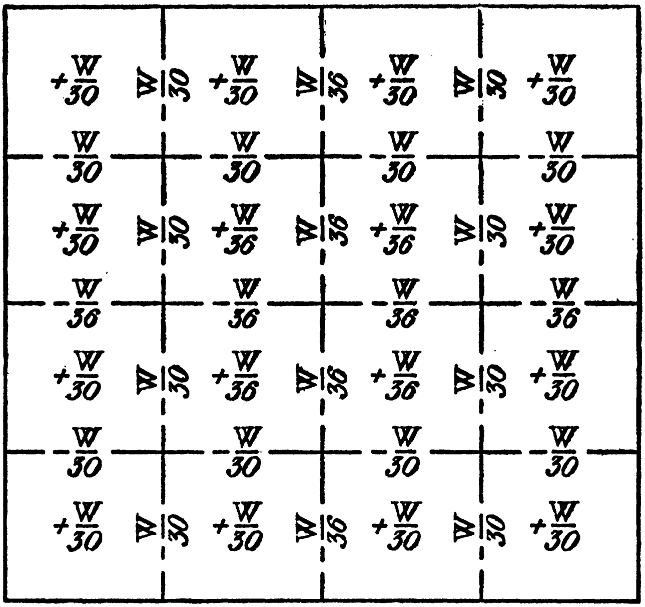

+W/30 W/30 +W/30 W/36 +W/30 W/30 +W/30
-W/30 -W/30 -W/30 -W/30
+W/30 W/30 +W/36 W/36 +W/36 W/30 +W/30
-W/36 -W/36 -W/36 -W/36
+W/30 W/30 +W/36 W/36 +W/36 W/30 +W/30
-W/30 -W/30 -W/30 -W/30
+W/30 W/30 +W/30 W/36 +W/30 W/30 +W/30

Fig. 167

Shear Forces under Distributed Loads.—The intensity of loading on the beams is greatest at a distance of about one quarter the span from each corner. In practice in continuous panels the total load on each panel may safely be taken as uniformly distributed along the perimeter (always assuming that the length never exceeds 1·5 times the breadth). The extreme corners tend to rise and this point should be watched when putting mesh panel floors on steel beams. For isolated mesh panels the supporting beams should be designed for :—

$$\frac{Wl}{2\,(l+b)\,6} \text{ and } \frac{W\,b}{2\,(l+b)\,6}$$

but for continuous panels the beams may be designed for :—

$$\pm \frac{W}{(l+b)}\,\frac{l}{12} \text{ and } \pm \frac{W}{(l+b)}\,\frac{b}{12}$$

where W is the total load on one panel and l and b are the long and short sides. The beams should be made as deep as possible to give stiff support to the panels.

Concentrated Loads.—The occurrence of these in practice is confined to wheel loads on bridge floors which may run over any part of the panel. We may assume the worst case is when the concentrated load is at the mid point. If we have an isolated square panel *r t u v* as in Fig. 168 carrying a concentrated load W_c spread over an area of $s \times s$ the *average* moment on section *vt* is $\frac{W_c}{24}\left(3-\frac{2s}{l}\right)$.

If $s = 0$ then this average moment is $\frac{W_c}{8}$.

Although it is very clearly established that the maximum moment under a distributed load at failure point is equal to the average moment, the author is not satisfied that a panel carrying a point load might not collapse due to some sort of progressive local failure. In face of this uncertainty he recommends the adoption of a moment of $\frac{W_c}{24}\left(5-\frac{4s}{l}\right)$ for designing purposes.

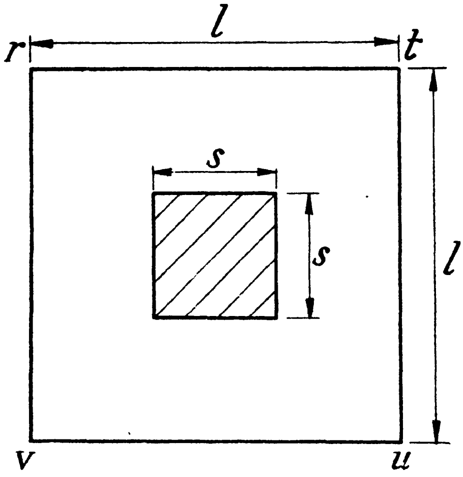


Fig. 168

If the concentrated load covers a rectangle we may reduce it to an equivalent square of the same area. If the panel is continuous over all four edges we may take ± two-thirds of $\frac{W_c}{24}\left(5-\frac{4s}{l}\right)$. If continuous over two or three edges take ± four-fifths.

For a knife-edge load W_k running right across the panel a moment of $\frac{W_k}{8}$ may be used for isolated panels, $\pm\frac{W_k}{10}$ for panels continuous over two or three, and $\pm\frac{W_k}{12}$ for panels continuous over four edges where W_k is the total knife-edge load on the panel.

Reinforcement in Mesh Panels.—The foregoing treatment of the moments takes the panel as a single unit and calculates the maximum ruling moment in any direction. Although this maximum moment in general occurs across the diagonals it is very awkward to dispose the reinforcement diagonally but if we supply equal rows of bars in both directions, the panel will have the same unit bending strength at all sections in all directions. This is exactly what we require and all panels whether square or rectangular should have the same unit reinforcement both ways.

In the centre of the panel one row of bars must lie on top of the other and we may take d as the average depth of the two rows. In the corners of fully continuous panels the bent-up bars in one direction must lie under the bent-up bars in the other direction, but over the middle half of each supporting beam the bent-up bars may come up high on each one of the four sides. Apart from these difficulties due to having two layers of bars, the steel will be arranged exactly as the steel for slabs spanning one way only, the bent-up bars running past the beams for a distance of one quarter of the span into all adjoining panels. Four short distributors are required in each fully continuous panel to tie together the ends of the bent-up bars.

Magnitude of Moments.—The unit moments are given in the form $\frac{W}{24}$, etc. If W is expressed in pounds the unit moments $\frac{W}{24}$, etc., are in lbs.-ft. per foot width of slab section.

Example.—A bridge floor is constructed as continuous mesh panels each 10 ft. 0 ins. × 12 ft. 0 ins. It is covered with 3 ins. of tarmac and has to carry a maximum wheel load of 11 tons (including impact). Two wheels may occur on one panel. Design one panel to stresses of 750 and 18,000 ($m = 18$).

The spread of the load and areas of contact are as in Fig. 169.

Area of contact $= 1' \, 0'' \times 5' \, 6'' = 5{\cdot}5$ sq. ft. $= 2{\cdot}35' \times 2{\cdot}35'$.

Area of panel $= 10' \, 0'' \times 12' \, 0'' = 120$ sq. ft. $= 10{\cdot}95' \times 10{\cdot}95'$.

We may take $\frac{s}{l}$ in Fig. 168 as $\frac{2{\cdot}35}{10{\cdot}95} = 0{\cdot}215$.

$$M = \pm \frac{2}{3} \times \frac{(22 \times 2240)}{24} \times \left(5 - 4(0{\cdot}215)\right) \text{ lbs.-ft. per 12 ins.}$$

width of section.

$$= \pm \frac{2}{3} \times \frac{22 \times 2240}{24} \times \left(5 - 4\,(0{\cdot}215)\right) \times 12 \text{ lbs.-ins. per 12 ins. width}$$

= 68,000 lbs.-ins. due to concentrated load alone.

Distributed load	= 10 × 12 × 40 lbs. =	4,800 lbs. tarmac.
plus say	10 × 12 × 102 lbs. =	12,200 lbs. self weight
		17,000 lbs.

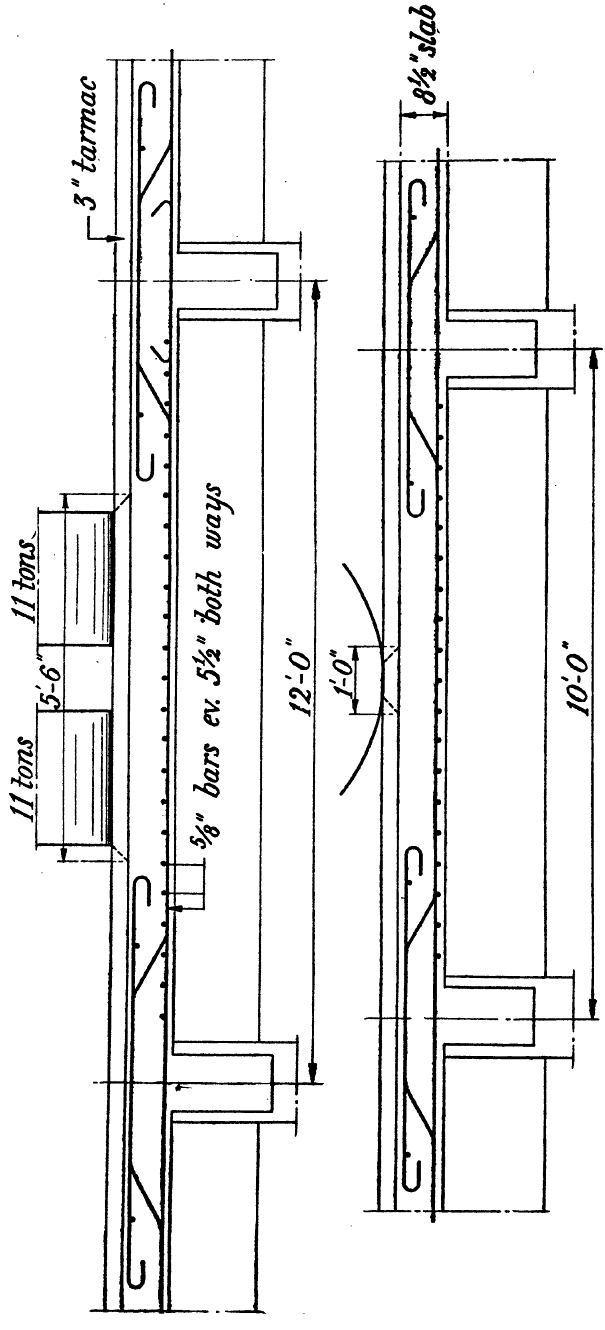

3" tarmac
11 tons
5'-6"
11 tons
5/8" bars ev. 5½" both ways
12'-0"
1'-0"
10'-0"
8½" slab

Fig. 169.

$$M = \pm \frac{W}{36} = \pm \frac{17{,}000}{36} \times 12 \text{ lbs.-ins. per 12 ins. width}$$

$$= 5700 \text{ lbs.-ins. due to dead distributed load.}$$

Total moment = 68,000 + 5,700 = 73,700 lbs.-ins.

$$\text{Min}^{\text{m}}.\ d \quad = \sqrt{\frac{73{,}700}{138 \times 12}} = 6{\cdot}66 \text{ ins.}$$

Use 8½-in. slab with ¾-in. cover and an average *d* of about 7 ins.

$$A_s = \frac{73{,}700}{18{,}000 \times 0{\cdot}86 \times 7} = 0{\cdot}675 \text{ sq. ins.}$$

say ⅝-in. bars every 5½ ins. both ways.

Quantities.—The quantities of concrete and shuttering per square yard are similar to those for slabs spanning one way (see Chapter XIII). The steel factor will be somewhat smaller as bars will be omitted on all four sides of the panel. A figure of (36 × 2) will cover nearly all cases of fully continuous slabs.

In the above example $A_s = 0{\cdot}67$ sq. ins. both ways.

The weight of steel per square yard of bridge floor is approximately 0·67 × 36 × 2 = 48 lbs. per sq. yd.

Drawings.—This requires a special note in connection with mesh panels. Not only should sections be drawn in both directions, but some kind of bar plan must be drawn, at any rate for the first few bays of floor to be laid, or the descriptive list will be difficult to make and very difficult to follow.

Application of Mesh Panels.—Mesh panel floors are not very common. Where the column spacing admits of large mesh panels a flat-slab design will generally give more satisfactory results. The fundamental idea of carrying load in both directions and thus halving the total shears on the main beams is discussed in the chapter on Continuous Footings (Chap. XXIX).

Interlacing Beams.—The treatment of interlacing systems of beams such as sometimes occur in the roof of a small circular tank will be fairly obvious. The moments for mesh panel slabs cannot apply to the case of isolated beams, as beam systems only intersect at points whereas the strength of a slab lies greatly in its torsional shears.

In cases where the deflections at all the beam crossings would be the same for each of the intersecting beams taken independently, then the interlacing beams may be looked upon as two mutually independent systems and treated as such. In all other cases the system should be looked upon as consisting of main and secondary beams and be designed accordingly.

CHAPTER XVIII

ARCHED MEMBERS

The author does not propose to give the elementary basis of deflection in thin curved members as this can be found set forth elsewhere.*

In all arch design we assume as a preliminary hypothesis that the abutments do not yield, and we have as a consequence the two following conditions :—

(a) The two points of abutment remain at the same height.

(b) The two points of abutment remain a fixed distance apart.

In cases of encastré arches which include the majority of small bridges, etc., we have a further condition.

(c) The slope at each point of abutment remains unchanged, i.e., the total change of slope in the whole member is nil.

These three conditions when expressed mathematically will give us the bending moment, the horizontal thrust and the vertical reaction at one abutment, which, with the statics of equilibrium, give a complete solution.

Assumptions.—It will be assumed that the arch has no internal stress when unloaded and that the deflections are caused wholly by bending effect. Adjustments to allow for temperature stresses and shortening due to direct stress will be introduced later. In the preliminary designs also we shall assume $\frac{ds}{EI\,dx}$ is a constant. The expression $E\,I$ is the same as that discussed at some length in Chapter XVI, where it is shown that theoretically the value $E\,I$ varies with the eccentricity of application of load for any section. Under a bending moment only, the value of $E\,I$ may be only 60% of its uncracked value. Leaving this aspect of the question for the moment, it will be seen that I must increase as $\frac{dx}{ds}$ diminishes, i.e., that the section of the arch must increase from the crown towards the springing. In cases of practical encastré arches it will be found that such an increase is always supplied, although not perhaps in strict proportion to $\frac{ds}{dx}$. As will appear later the conditions of loading, support and deflection in any actual arch are so vague that errors introduced by writing $\frac{ds}{EI\,dx} =$ a constant, are probably negligible.

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* See for example Morley's *Strength of Materials* or *Theory of Structures.*

(The author is unaware of the originator of this assumption but would like to acknowledge his indebtedness.)

The most popular shape for small arches is the parabola and this will be given first treatment.

General Method.—The arch will be supposed to carry a single rolling load and the moments, etc., will be calculated for any position of this load, thus giving us influence lines similar to those obtained in Chapter X. These will be integrated where necessary to give envelopes for superloads. The arch will be assumed split and turned upside down as this allows us more easily to treat cases of unsymmetrical arches which sometimes occur. This treatment divides the arch into two separate cantilevers. It must be clearly understood that this method can be applied to encastré arches only and is used here partly to simplify the integration and partly to give the reader an insight into both methods. In calculating arches with more complex profiles and varying moments of inertia it allows the use of odd functions of x. In calculating the two semicircular arches and the hinged parabola the springing line is taken as axis of x and one hinge as origin. This method is necessary in two-hinged arches as we cannot divide them into two cantilevers. Comparing the two methods we see that the transference of the origin is permissible in encastré arches because of the condition

$$\int \frac{M}{EI}\, d\, s = 0.$$

For if we write :—

$$\int \frac{M\,(x-c)}{E\,I}\, d\, s \text{ instead of } \int \frac{M\,x}{E\,I}\, d\, s$$

where c is a constant

we have $$\int \frac{M(x-c)}{E\,I}\, d\, s = \int \frac{M\,x}{E\,I}\, d\, s - c \int \frac{M}{E\,I}\, d\, s$$

$$= \int \frac{M\,x}{E\,I}\, d\, s + \text{zero.}$$

Similarly we may move the axis from which y is measured. In hinged arches the condition

$$\int \frac{M}{E\,I} d\, s = 0$$

does not hold and the origin cannot be moved. In the analysis no effort is made to distinguish between upward and downward deflection as the sign makes no difference, the total being zero.

Analysis.—Parabolic Encastré Arch.

Let $2\,a$ = span
b = rise
W = point load
k = a fraction (variable).

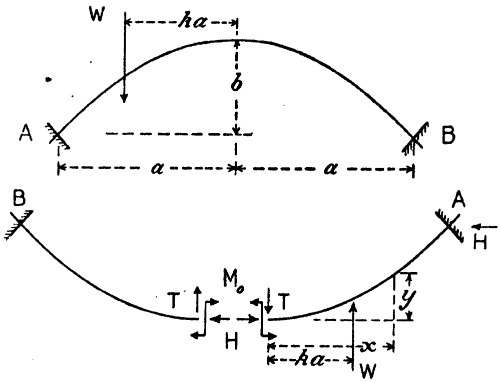


Fig. 170.

Take half the span and invert it as shown in Fig. 170. This half member is held in equilibrium by

a horizontal thrust = H,

a bending moment M_o,

a vertical reaction T all of which are caused by reactions from the other half member. Also a load W and fixing moment and reactions at the abutment A. Since the applied load W acts vertically the value of H is the same at all sections of the member for any one position of W. The value of T will be constant over unloaded portions of the member. M_o is to be reckoned positive when causing tension on the intrados.

Take any point (x, y) and let the bending moment at that point be M.

From the equation of a parabola we have

$$y = (\text{constant}) \times x^2$$

as $y = b$ when $x = a$

$$b = \text{constant} \times a^2$$

$$\text{constant} = \frac{b}{a^2}.$$

For convenience write $\frac{b}{a^2} = c$, i.e., $y = cx^2$

The bending moment at any point of the right-hand half-member is given by :—

$$M = M_o + Tx + Hy - W(x - ka).$$

For values of x which are less than $k\,a$ the last term vanishes the bending moment diagram being discontinuous at the point $x = k\,a$. For purposes of analysis we may assume that the bending moment diagram consists of two parts. One part is given by

$$M = M_o + Tx + Hy$$

over the whole half-member. The other part is given by :—

$$M = -W(x - ka) \text{ from } x = ka \text{ to } x = a.$$

By well-known formulae we have

$$\text{Change of slope over member} = \int \frac{M}{EI}\,ds$$

$$\text{Vertical deflection} = \int \frac{M}{EI}x\,.\,ds$$

$$\text{Horizontal deflection} = \int \frac{M}{EI}y.\,.\,ds.$$

Taking the first of these and applying it to our half-member

$$\int \frac{M}{EI}\,ds = \frac{1}{E}\int_0^a \frac{(M_o + Tx + Hy)}{I}\,ds - \frac{1}{E}\int_{ka}^{a} \frac{W(x - ka)}{I}\,ds.$$

Now $\frac{ds}{EI\,dx}$ = constant = q say, i.e., $\frac{ds}{EI} = q\,.\,dx$

$$\int \frac{M}{EI}\,ds = q\int_0^a (M_o + Tx + Hy)\,dx - q\int_{ka}^{a} W(x - ka)\,dx.$$

This represents the change of slope on the loaded half. By a similar process for the unloaded half we have

$$\text{Change of slope} = q\int_0^a (M_o - Tx + Hy)\,dx.$$

Since the arch is encastré at the abutments the total change of slope is nil

$$\text{i.e.,}\quad q\int_0^a (M_o + Tx + Hy)\,dx - q\int_{ka}^{a} W(x - ka)\,dx + q\int_0^a (M_o - Tx + Hy)\,dx = 0$$

$$2\int_0^a (M_o + Hy)\,dx = \int_{ka}^{a} W(x - ka)\,dx$$

$$2M_o a + 2Hc\frac{a^3}{3} = W\left[\frac{x^2}{2} - kax\right]_{ka}^{a}$$

$$2M_o + 2Hc\frac{a^2}{3} = \frac{Wa}{2}(1 - 2k + k^2) \quad \ldots\ldots\ldots \quad (1)$$

Coming now to the vertical deflection we have

$$q\int_0^a (M_o + Tx + Hy)\,x\,dx - q\int_{ka}^{a} W(x - ka)\,x\,dx$$

for one half must equal

$$q\int_o^a (M_o - T\,x + H\,y)\,x\,d\,x \text{ for the other half.}$$

Which gives us :—

$$2\int_o^a T\,x^2\,d\,x = \int_{ka}^a W\,(x - k\,a)\,x\,d\,x$$

$$T = \frac{W}{4}\,(2 - 3\,k + k^3) = W\left(\frac{1}{2} - \frac{3}{4}k + \frac{k^3}{4}\right) \quad . \quad . \quad . \quad (2)$$

The sum of the horizontal deflections for the two halves must be zero giving us :—

$$q\int_o^a (M_o - T\,x + H\,y)\,y\,d\,x + q\int_o^a (M_o + T\,x + H\,y)\,y\,d\,x$$

$$-\,q\int_{ka}^a W\,(x - k\,a)\,y\,d\,x = 0$$

$$2\int_o^a (M_o + H\,y)\,y\,d\,x = \int_{ka}^a W(x - k\,a)\,y\,d\,x$$

$$2\,M_o + \frac{6}{5}c\,H\,a^2 = 3\,W\,a\left(\frac{1}{4} - \frac{k}{3} + \frac{k^4}{12}\right) \quad . \quad . \quad . \quad (3)$$

Equation (2) gives a value for T directly while equations (1) and (3) will give values of M_o and H.

We have :—

$$2\,M_o + 2\,H\frac{c\,a^2}{3} = Wa\left(\frac{1}{2} - k + \frac{1}{2}k^2\right). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$2\,M_o + \frac{6}{5}H\,c\,a^2 = Wa\left(\frac{3}{4} - k + \frac{k^4}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$H\,c\,a^2\left(\frac{18 - 10}{15}\right) = Wa\left(\frac{1}{4} - \frac{1}{2}k^2 + \frac{1}{4}k^4\right) \text{ by subtraction.}$$

Now $ca^2 = b$

$$\therefore \quad H = \frac{Wa}{b}\left(\frac{15}{32} - \frac{15}{16}k^2 + \frac{15}{32}k^4\right) \quad . \quad . \quad . \quad . \quad (4)$$

Putting this value of H in equation (1)

$$2\,M_o = Wa\left(\frac{1}{2} - k + \frac{1}{2}k^2\right) - Wa\left(\frac{5}{16} - \frac{5}{8}k^2 + \frac{5}{16}k^4\right)$$

$$M_o = Wa\left(\frac{3}{32} - \frac{1}{2}k + \frac{9}{16}k^2 - \frac{5}{32}k^4\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By writing $k = \cdot 2$, $k = \cdot 4$, $k = \cdot 6$, etc., we have values for T, H and M_o. The bending moments at all other points may be found by simple statics.

If $k = \cdot 4$

$$T = W\left(\frac{1}{2} - \cdot 3 + \cdot 016\right) = \cdot 216\,W$$

$$H = \frac{Wa}{b}(\cdot 469 - \cdot 15 + \cdot 012) = \cdot 331\,W\frac{a}{b}$$

$$M_o = Wa\,(\cdot 094 - \cdot 2 + \cdot 09 - \cdot 004) = -\,\cdot 020\,Wa.$$

From Fig. 170 it follows that

$$\begin{aligned} M_A &= -\,\cdot 020\,Wa + \cdot 331\,Wa + \cdot 216\,Wa - \cdot 60\,Wa \\ &= -\,\cdot 073\,Wa \\ M_B &= -\,\cdot 020\,Wa + \cdot 331\,Wa - \cdot 216\,Wa \\ &= +\,\cdot 095\,Wa. \end{aligned}$$

The bending moment under the load is

$$\begin{aligned} &-\,\cdot 020\,W\,a + \cdot 216 \times \cdot 4\,Wa + \cdot 331 \times \cdot 16\,Wa \\ &= (-\,\cdot 020 + \cdot 086 + \cdot 053)\,Wa \\ &= +\,\cdot 119\,Wa \end{aligned}$$

We see that the values of the bending moments and thrusts are independent of the actual value of the moment of inertia provided the relation $\frac{d\,s}{I\,d\,x}$ = constant is adhered to. When we consider temperature stresses this independence ceases.

The results of a distributed load covering one-half the span can be obtained by writing

$W = \delta\,(ka)\,w = a\,(\delta k)\,w$ and integrating whence

$$T = \int_0^1 wa\left(\frac{1}{2} - \frac{3}{4}k + \frac{1}{4}k^3\right)d\,k$$

$$= \frac{3}{16}w\,a.$$

The effect of loading the other half-span would be to increase T by $-\frac{3}{16}\,w\,a$ as will be seen from Fig. 170. Thus when the whole span is covered by a uniform load,

$$H = 2\int_0^1 w\,a \times \frac{a}{b}\left(\frac{15}{32} - \frac{15}{16}k^2 + \frac{15}{32}k^4\right)d\,k$$

$$= 2\,w\,a\left(\frac{1}{4}\right) \times \frac{a}{b} = \frac{w\,a^2}{2\,b}$$

$$M_o = 2\int_0^1 w\,a^2\left(\frac{3}{32} - \frac{1}{2}k + \frac{9}{16}k^2 - \frac{5}{32}k^4\right)d\,k$$

$$= 0.$$

These results show, as might have been expected, that a uniform load covering the whole span causes no bending moment anywhere in the arch, the line of resultant thrust following the centre line of the arch rib. The maximum possible bending moment which can be produced at the crown by a distributed load is obtained by loading the positive area of the influence line in Table A II. The point where the influence line cuts the zero line is found by finding the points where we may place a point load so that it produces no bending moment at the crown

$$\text{i.e.,}\quad M_o = Wa\left(\frac{3}{32} - \frac{1}{2}k + \frac{9}{16}k^2 - \frac{5}{32}k^4\right) = 0$$

giving

$$\begin{aligned} k &= 1 \\ k &= 1 \\ k &= -\,2{\cdot}262 \\ k &= +\,{\cdot}262. \end{aligned}$$

Maximum value of M_o under a superload

$$= 2\int_o^{\cdot 262} wa^2\left(\frac{3}{32} - \frac{1}{2}k + \frac{9}{16}k^2 - \frac{5}{32}k^4\right)dk$$

$$= +\,{\cdot}0219\,w\,a^2.$$

The value of H corresponding to this is :—

$$2\int_o^{\cdot 262} w\,a \times \frac{a}{b}\left(\frac{15}{32}k - \frac{5}{16}k^3 - \frac{3}{32}k^5\right)$$

$$= {\cdot}235\,w\,a \times \frac{a}{b}.$$

The vertical reaction at each abutment will be, by symmetry, $\cdot 262\,w\,a$.

Bending Moment Envelope for Uniform Superload.—If we take each influence line in Table A II in turn and spread a uniform load over the positive area only, we have values for the greatest possible positive moments at all points of the arch. By drawing these values out we have an envelope of moment. That is to say that all positive bending moments in the arch caused by any and every possible distribution of superload will lie inside this curve. By taking the negative areas of each influence line in turn and loading these areas with superload, we have a similar envelope for negative moments. Both these curves are given in Fig. 171 which also shows the horizontal thrusts corresponding to the loadings which cause maximum positive and maximum negative moments.

Having now completely solved the problem based on our original assumptions we must find what modifications are necessary for temperature stresses.

Temperature Stresses.—Suppose for a moment that the arch is free from all loads and from all reactions from the supports. Suppose that a rise in temperature occurs equal to R.

Let E = elastic modulus

Φ = coefficient of expansion

I_o = moment of inertia at the crown.

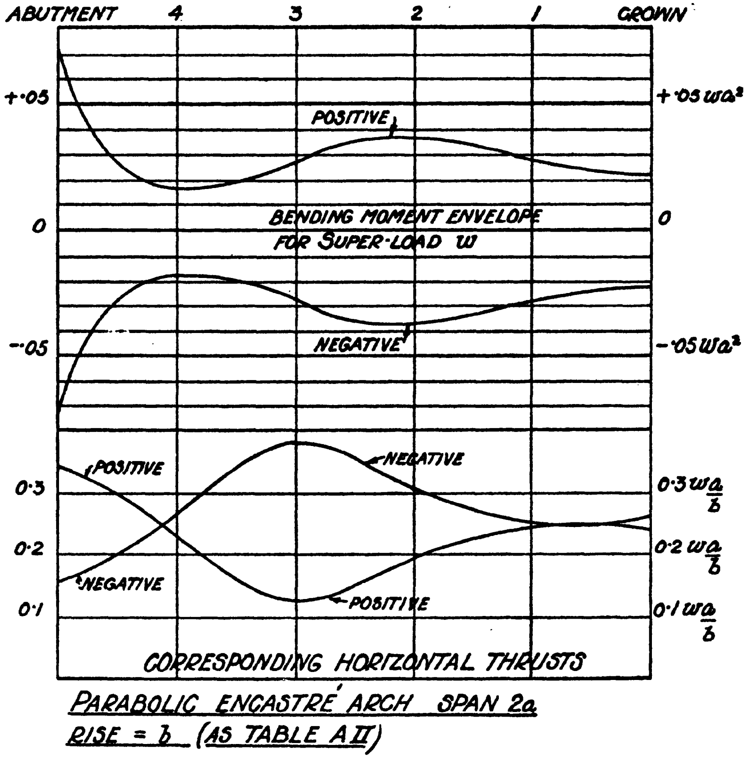


Fig. 171.

The rib would expand in all directions the height b would increase to $b\,(1 + \Phi\, R)$ and the length $2\,a$ would increase to $2\,a\,(1 + \Phi\, R)$. If now we apply thrusts and moments at both ends such that the length is forced back to its original value $2\,a$ and the tangents at the ends are made to line with their original directions, we shall have the conditions obtaining when the arch suffers a rise in temperature and the abutments do not yield. It appears, from the conditions that the arch is unloaded by vertical loads, that the vertical shear is nil.

Referring to our original equation for the horizontal deflection at the crown.

Deflection of half arch (shortening)

$$= q \int_0^a (M_o + H y)\, y\, d\, x$$

which must equal $a\,\Phi\, R$.

And since the tangents are held at the abutments and the tangent at the crown is, by symmetry, horizontal, the change of slope on the half arch is zero

$$\text{i.e.,}\quad q \cdot \int_o^a (M_o + Hy)\,dx = 0$$

giving $$2M_o + 2Hc\frac{a^2}{3} = 0 \text{ as in equation (1)}$$

$$M_o = -\frac{Hb}{3}.$$

Solving $$q\int_o^a (M_o + Hy)\,y\,dx = a\Phi R$$

after the manner of equation (2)

$$q \times 2c\left(M_o\frac{a^3}{3} + \frac{Hca^5}{5}\right) = 2a\Phi R$$

$$\left(M_o \cdot \frac{b}{3} + \frac{Hb^2}{5}\right)a = \frac{a\Phi R}{q}$$

as $$M_o = -\frac{Hb}{3}$$

$$-\frac{Hb^2}{9} + \frac{Hb^2}{5} = \frac{\Phi R}{q}.$$

Now $q = \frac{ds}{EIdx}$ and when $x = 0$, $q = \frac{1}{EI_o}$ (at the crown)

$$\therefore\quad H = \frac{45EI_o\Phi R}{4b^2} \text{ and } M_o = -\frac{15EI_o\Phi R}{4b}.$$

If we write E = 2,000,000 lbs. per sq. in., I_o being reckoned in concrete units and Φ = ·000006, R being in degrees Fahrenheit

$$\left.\begin{aligned} H &= \frac{135I_oR^\circ}{b^2} \\ Mo &= -\frac{45I_oR^\circ}{b} \end{aligned}\right\}\quad \begin{gathered} I_o \text{ in inch units (concrete)} \\ b \text{ in inches.} \end{gathered}$$

The results will be H in lbs. and M_o in lbs.-ins. (See Table A IV.)

The readjustment of stress due to the shortening of the arch by direct thrust, which we have so far neglected, is very similar to the effect caused by a fall in temperature. If the arch ring were under a direct compression of 400 lbs. per sq. in. at all sections the shortening of the span would be

$$\frac{400}{2{,}000{,}000} \times 2a = \cdot 0002 \times 2a.$$

Under a temperature fall of 30° F. the shortening would be

$$30 \times \cdot 000006 \times 2a = \cdot 00018 \times 2a.$$

Both effects for normal ratios of span to rise are fairly small.

In English practice it is usual to allow a temperature difference of + 30° F. corresponding to a total range of 60° F. No allowance is made for the fact that an arch concreted with the temperature of 40° might reach a summer heat of 90° and vice versa.

Stresses due to shrinkage of concrete during hardening are exactly similar to those caused by a fall in temperature.

Tables A I and A II.—These two tables show in detail the different thrusts and moments which follow from our analysis.

Semi-Elliptical Arch.—Analysis.

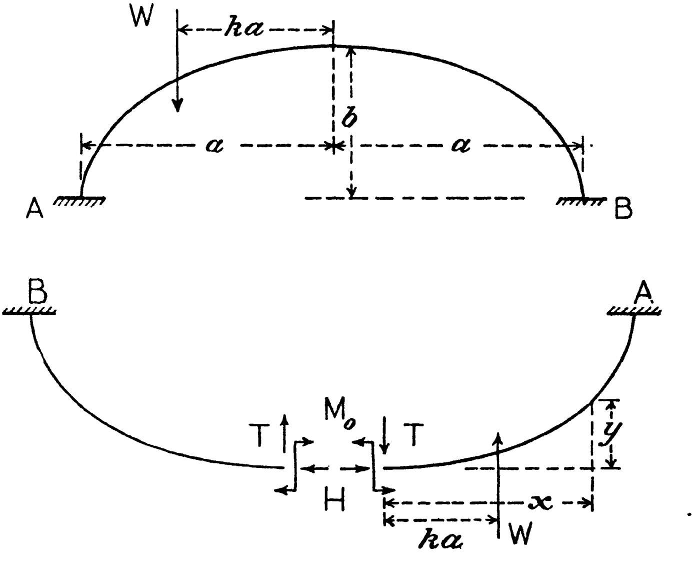


Fig. 172.

The equation of an ellipse referred to the centre line and springing-line as axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2\left[1 - \frac{x^2}{a^2}\right]$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Turning the arch upside down and taking the origin to the crown we have :—

$$y = \left(b - \frac{b}{a}\sqrt{a^2 - x^2}\right)$$

Let M_o = moment at crown

H = horizontal thrust

etc., as for the parabolic arch.

Write $\frac{d\,s}{E\,I\,d\,x} = q$, a constant.

This makes I infinite when $\frac{d\,s}{d\,x}$ becomes infinite which condition may be more or less realized practically by running the arch into the abutment shortly before the theoretical springing line is reached.

Omitting some of the steps in the analysis as the previous case has been given in full

$$M_o + \cdot 2146\, H\, b = Wa\left(\frac{1}{4} - \frac{k}{2} + \frac{k^2}{4}\right) \quad . \quad . \quad . \quad . \quad (6)$$

$$T = W \times \frac{3}{2}\left(\frac{1}{3} - \frac{k}{2} + \frac{k^3}{6}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\cdot 4292\, M_o + \cdot 1918\, H\, b = W\,a\left[\frac{1}{2} - \cdot 2146\, k + \frac{k^2}{2} - \frac{(1-k^2)^{3/2}}{3} - \frac{k^2}{2}\sqrt{1-k^2} - \frac{k}{2}\sin^{-1} k\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Eliminating H from equations (6) and (8)

$$M_o = \frac{Wa}{\cdot 0997}\Big[-\cdot 05935 - \cdot 04986\, k - \cdot 05935\, k^2 + \cdot 0715\,(1-k^2)^{3/2} + \cdot 1073\, k^2\sqrt{1-k^2} + \cdot 1073 \sin^{-1} k\Big]$$

If we put $k = 0$ we have $M_o = + \cdot 122\, Wa$ against $+ \cdot 094\, Wa$ for the parabolic arch.

If we put $k = 1$

$$M_o = \frac{aW}{\cdot 0997}\,[\cdot 16856 - \cdot 16854] = 0$$

within our limits of working, thus checking the analysis.

Substituting the calculated values of M_o in equation (6) which is

$$H = \frac{a}{b}\left[\frac{W(1-k^2) - \frac{4\,M_o}{a}}{0\cdot 8584}\right]$$, we get the values of H.

Temperature Stresses in Semi-Elliptic Arch.—Using the same notation as before

$$M_o + 0\cdot 2146\, H\, b = O$$

and

$$H = \frac{E\,I_o\,\Phi\,R}{\cdot 04985\, b^2}$$

The vertical reactions at the supports are identical in the two cases and each equal to the reactions occurring in a simple encastré beam of span 2 a having $E\,I$ = a constant.

The detailed results of the analysis for a semi-elliptic arch are set out on Tables A III, A V and A VI.

Conditions in Actual Arches.—In bridges where the roadway consists of a floor resting on columns which are in turn supported on arched ribs, the conditions of loading are fairly definite and, provided the abutments are sufficiently massive, there seems a fair chance that the analysis will follow the stresses. Great care must be taken to ensure that the falsework is rigid and that concreting is carried out as continuously as possible having regard to the fact that the centering must be fairly evenly loaded. In small bridges consisting of an arched sheet with reinforced concrete spandrel walls where the spandrels are filled in, the loading under point loads becomes very vague. If the bridge is narrow compared with its span then the conditions can only be guessed at. In such cases the greater part of the load is carried by the spandrel walls acting as continuous double cantilevers and the structure is not really an arch at all. It is standard practice, however, to design all such cases as arches as the quantities in a small arch sheet are not very large.

Yielding of Abutments.—It is not sufficient that abutments should be merely stable. They must be sufficiently massive to be incompressible. Abutments which rely largely on counter-pressure from earth backing should be avoided. Using a factor of safety of two we can rely on earth to resist a horizontal thrust of $\frac{1}{2} \times wh^2 \frac{(1 + \sin\theta)}{(1 - \sin\theta)}$ as in Rankine's formula, but in taking up this pressure the backing must be compressed and consolidated horizontally and may yield appreciably under the process. The effect of yielding of the abutments is very similar to that of a fall in temperature. In order to get a rough working idea of the magnitude of this effect let us take a parabolic arch of 35 ft. 0 ins. span and 7 ft. 0 in. rise consisting of an arched sheet say 8 ins. thick at the crown.

$$E\,I_0 = \text{about } 2{,}000{,}000 \times 1{\cdot}5 \times 12 \times \frac{8 \times 8 \times 8}{12} \text{ lbs.-ins.}^2$$

for a width of 12 ins.

From our analysis of the temperature stresses :—

$$-H = \frac{45}{4} \times \frac{(\text{increase in length})}{2\,a\,.\,q\,.\,b^2}$$

$$H = -\frac{45}{8}\,.\,\frac{E\,I_0}{a\,b^2}\,(\text{increase in length}).$$

Now suppose that each abutment yields an amount of $\frac{2\,a}{1000}$ i.e., one thousandth of the span (0·42 ins.).

$$\text{Total yield} = \frac{4\,a}{1000} = \frac{2 \times \text{span}}{1000} \text{ (corresponds to a fall of 333° F.)}$$

$$H = -\frac{45}{8}\frac{E\,I_o}{ab^2} \times \frac{4\,a}{1000} = -\frac{45}{2000} \times \frac{E\,I_o}{b^2}$$

$$= -\frac{45}{2000} \times \frac{2{,}000{,}000 \times 1{\cdot}5 \times 8 \times 8 \times 8}{84 \times 84} = -\ 4910 \text{ lbs.}$$

$$M_o = -\frac{1}{3}\,Hb = \frac{4910}{3} \times 84 = 138{,}000 \text{ lbs.-ins.}$$

The rise we have taken is one-fifth the span and even this gives a bending moment at the crown far in excess of our safe strength. For a rise of one-tenth the span the thrust due to each abutment yielding one-thousandth of the span would be four times as large and the moment at the crown twice as large as above. We see therefore that in order to keep the abutments incompressible from a practical stand-point neither must yield more than about say one-twenty-thousandth of the span. In the case quoted the allowable yield would be less than $\frac{420}{20{,}000} = {\cdot}021$ in. each in order to be negligible, which is a very small amount.

The ratio of span to rise should never exceed 10 to 1.

For small arched sheets having monolithic spandrel walls provided that the width does not exceed about half the span, a ratio of 10 to 1 may be employed if headroom is very scarce. If the designer is free to choose a ratio he should take between 5 to 1 and 4 to 1. Semi-elliptical or five-centre arches will only be used for small ratios of span to rise say 4 to 1 up to 2 to 1 when the arch becomes semi-circular.

Conditions of Loading.—Where the spandrels are filled in with earth as is generally the case for small road bridges the determination of stresses caused by isolated wheel loads becomes difficult. Some designers assume an area of contact of the wheel with the road surface say 15 ins. × 3 ins. The load is then assumed to " spread " at 45° down to the tensile steel in all directions giving a small square-shaped patch of loaded area. A strip of arch 12 ins. wide is chosen running through this loaded area and designed to carry the load acting as an independent arched rib. There can be no doubt that a point load is spread sideways by a continuous arched sheet in the same way as it would be spread by a slab (see Chapter XIII) although the author knows of no reliable data on the subject. Basing our ideas on the spread in a slab (which spread has been verified by actual measurement) it would seem that the axle load on a highway bridge may be assumed as spread out on a line 10 ft. 0 ins. long at the road surface and may be assumed to " spread " through the filling at 60° to the vertical down to the centre-line. The importance of spread in the direction of the span will appear from an inspection of Table A II. The bending moment influence line for any point comes up to a sharp cusp at that point. The average ordinate therefore for a loaded area is considerably less than the " peak " ordinate for a point load.

The impact allowance for an arch with earth-filled spandrels may be taken as 25% for road vehicles and 50% for railway traffic as the filling undoubtedly steadies the whole structure against shocks.

When designing public road bridges the requirements of the Ministry of Transport will probably have to be studied.

Application of Influence Lines (spans up to 50 ft. 0 ins.)—Most bridges of this size are built as arched sheets with the spandrels filled in with hard filling or hardcore. When confronted with the problem of designing such an arch the following procedure may be followed. It is assumed that the span and rise have been fixed by the conditions existing at the site and by the general rules governing the design of structures (see Part III). Unless there are special local reasons, we may adopt a parabolic profile (it is, of course, the centre-line of the arch and not the soffite which is parabolic). For a road bridge 60 ft. 0 ins. span a crown thickness of 12 ins. is generally sufficient and for 40 ft. 0 ins. span about 9 ins. Bridges of 35 ft. 0 ins. span or less may be made 8 ins. thick at the crown. The thickness at the springing may be taken as one-and-a-half times the thickness at the crown. There is little object in making small bridges less than 8 ins. at the crown. An extra inch of thickness over the whole of an arched sheet 35 ft. 0 ins. span and 40 ft. 0 ins. wide would amount to less than 5 cub. yds. of concrete and would cost only about ten pounds to fifteen pounds. For small bridges the thickness may be made to vary uniformly from crown to springing. The profile of the arch is then drawn out to some convenient scale say ½-in. or ¼-in. to 1 ft. and the road level is drawn on it, thus showing the amount of earth filling to be carried. The arch is next divided into ten equal horizontal divisions by vertical lines. Fig. 173 shows an example. The dead load between each pair of vertical lines is next estimated (the analysis being taken for a 12-in. width of the arch sheet). The influence line for the section at the crown is then drawn using the values in Table A I. This is drawn to the same horizontal scale as the profile of the arch and is drawn immediately below it. The influence line for horizontal thrust is drawn immediately below again, the actual values of *a* and *b* being used, everything being kept in pounds and feet. The dead load of each of the ten divisions is assumed to act through the centroid of each division and vertical lines are drawn down through both influence lines. The vertical heights intercepted by each influence line give the bending moment at the crown and horizontal thrust due to the weight of each section of dead load. These are added up and give the total dead load effect.

We next come to the live load. If this consists of a train of wheels the vehicle having the heaviest wheel load should be placed with its heavily loaded wheels over the crown. This gives the maximum positive moment at the crown section. The wheel loads may be assumed as spreading at 60° down to the centre-line of the arch, thus giving a small area of distributed load instead of a point load. The sideways spread is then calculated and the loads per square foot on the arch obtained. The loaded areas on the influence lines are taken off and multiplied by the loads per square foot, giving the moment at the crown and the corresponding thrust. To obtain the worst negative moment at the crown section we should place one heavy wheel load between sections II and III and another between sections 2 and 3 as the bending moment influence line at the crown has two negative areas.

We next calculate the temperature stresses and the stresses due to shrinkage. The dead load, live load, temperature and shrinkage stresses are now combined and a correction added for crown-drop stresses (that is the shortening of the arch under direct thrust). We have two cases to consider for the crown section. First with dead load, shrinkage, maximum positive live load, fall in temperature and crown-drop corresponding to the combined thrusts. Secondly, with dead load, shrinkage, maximum negative live load, rise in temperature and corresponding crown-drop.

The section at the springing is next taken and the bending moment influence line, the horizontal thrust influence line, and the shear influence line are drawn. The dead load moment and thrust are taken from these influence lines and checked back by arithmetic against the values already obtained for the crown section. The live load is disposed to give the worst positive and the worst negative moments and these are combined with the dead load, the temperature and shrinkage stresses.

The sections are checked by reference to the tables in Chapter VIII. It should be noted that the influence lines give the horizontal thrust and the vertical shear at the springing section and these must be resolved tangential to, and at right-angles to, the arch centre-line to give the true thrust and the true shear on the springing section. In preliminary designs the crown and springing sections only need be examined. In final designs one or two other sections may be checked before detailing the steel.

Maximum Thrust on Abutment.—The maximum stresses at any section of the arch occur when the section carries its maximum bending moment but the worst loading, as far as the abutment is concerned, is generally when the whole bridge is loaded and the horizontal thrust is a maximum. Nevertheless it is as well to examine the stability of the abutment when the arch is loaded to give maximum positive moment at the springing and when the arch is loaded to give maximum negative moment at the springing.

Example.—An arch is built as a parabolic arched sheet with spandrels filled with hardcore. It is 35 ft. 0 ins. span with a rise of 6 ft. 0 ins. Find the necessary reinforcement at the crown section if this is 8 ins. thick. The bridge is to carry the Ministry of Transport Standard Load for Highway Bridges (Equivalent Loading Curve) 1931. The concrete will take a safe stress of 750 lbs. per sq. in. with $m = 15$.

There is 9 ins. of road metal over the crown. Allow a variation of ± 30° Fah. and a shrinkage of 0·02%.

As the arch is 35 ft. 0 ins. span a thickness of 8 ins. at the crown should be ample. We first require to assess the dead loads and we may assume that the arch thickness increases uniformly from 8 ins.

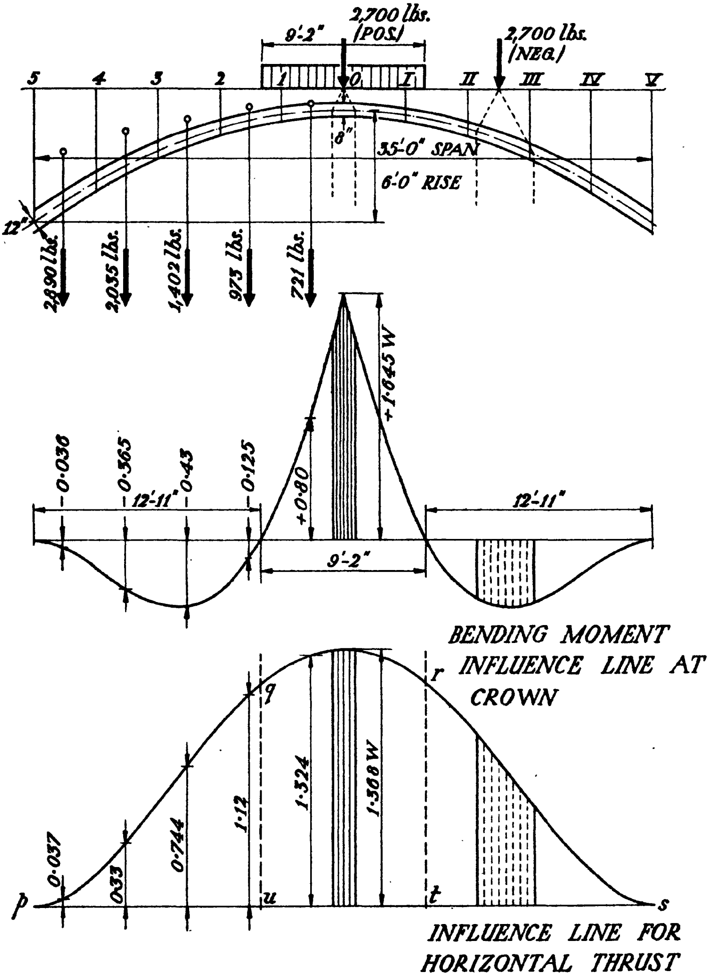


Fig. 173.

to 12 ins. at the springing. We may assume that the road metal and hardcore filling weigh 120 lbs. per cu. ft. and the reinforced concrete 144 lbs. per cu. ft.

A longitudinal section of the bridge is shown in the upper part of Fig. 173. Dividing this into ten equal horizontal lengths of 3 ft. 6 ins. we have ten divisions corresponding to those in Tables A I and A II, and we may mark them 5, 4, 3, 2, 1, 0, I, II, III, IV and V. The total weight of each division is given on the figure. The values for drawing the moment influence line for the crown section are given in the vertical column marked "CROWN" in Table A I. For a load W placed exactly over the crown the moment at the crown is $+ 0{\cdot}094\, W a$ where a is half the span

$$+ 0{\cdot}094\, Wa = + 0{\cdot}094\, W \times 17{\cdot}5 \text{ feet.}$$
$$= + 1{\cdot}645\, W \text{ foot units.}$$

If W is expressed in pounds then this moment is in pounds-feet. Multiplying the other values in the same column by 17·5 we have the whole influence line. This is drawn below the arch in Fig. 173. Also from Table A I the horizontal thrust caused by a load W at the crown is $0{\cdot}469\, W \frac{a}{b}$ where b is the rise

$$0{\cdot}469\, W \frac{a}{b} = 0{\cdot}469\, W \times \frac{17{\cdot}5}{6} = 1{\cdot}368\, W.$$

Taking all the values in the vertical column we have the thrust influence line. This is drawn in the lower part of Fig. 173. We next draw verticals through the centroid of each division and measure the intercepts on the two influence lines. Thus the section 0—I weighs 721 lbs. Drawing a vertical down through the centroid of this section, the line has an intercept of $+ 0{\cdot}80\, W$ on the moment influence line and an intercept of $1{\cdot}324\, W$ on the thrust influence line. The dead weight of this section therefore causes a moment of $+ 0{\cdot}80 \times 721 = + 577$ lbs.-ft. at the crown and a thrust of $1{\cdot}324 \times 721 = 955$ lbs.

Putting in the weights and intercepts of all sections we have

	Total M_o (dead).	Total H (dead)
5—4 and IV—V	− 208 lbs.-ft.	234 lbs.
4—3 and III—IV	− 1290	1340
3—2 and II—III	− 1210	2100
2—1 and I—II	− 241	2170
	− 2949	
1—0 and 0—I	+ 1154	1910
	− 1795 lbs.-ft.	7754 lbs.

We next take the live load. To obtain the maximum positive live load moment at the crown we must load the positive area of the moment influence line in Fig. 173 as heavily as possible and leave the negative areas unloaded. The length of the positive area is 9 ft. 2 ins. (We have calculated this length which is 0·262 times the span.) This constitutes the "loaded length" and the Equivalent Loading Curve calls for a uniform load of about 295 lbs. per sq. ft. over this length of 9 ft. 2 ins. and a knife-edge load of 2700 lbs. placed at the worst point. (This is obviously immediately over the crown.) We may assume that this knife-edge load " spreads " at 60° down to the arch centre-line and thus comes on the arch as a short loaded area. The advantage of spreading the load is clear from Fig. 173. The extreme height of the moment influence line at the point is 1·645 *W*. By spreading the load over a length of 1 ft. 3 ins. the average vertical height of the area shown shaded in the figure is 1·49 *W*. The moment at the crown due to a knife-edge load of 2700 lbs. is, allowing for spread, 2700 × 1·49 lbs.-ft.

The effect of a distributed load of 295 lbs. per sq. ft. over a length of 9 ft. 2 ins. covering the positive area of the moment influence line is 295 times the positive area. We may scale this area from the curve but we have previously calculated this value as $+ 0{\cdot}0219\, wa^2$ or $+ 0{\cdot}0219 \times 295 \times 17{\cdot}5^2$ lbs.-ft. (Alternatively the average height of the positive part of the influence line in Fig. 173 is 0·73 and the moment is 295 × 0·73 × 9·17 = 1980 lbs.-ft.) The thrust caused by the knife-edge load is 2700 multiplied by the average ordinate of the shaded area of the thrust influence line or 2700 × 1·36 lbs. The thrust due to a load of 295 lbs. per sq. ft. spread over a length of 9 ft. 2 ins. at the crown is 295 times the area *q r t u* of the thrust influence line immediately below it. We may scale the average height from the diagram but we have already calculated the thrust as $0{\cdot}235 \dfrac{w\,a^2}{b}$ or $0{\cdot}235 \times 295 \times \dfrac{17{\cdot}5^2}{6} = 3550$ lbs.

To subject the arch to the greatest possible negative live load moment at the crown we must load both sides from the springing for a length of 12 ft. 11 ins. leaving the centre part unloaded and in addition apply *one* knife-edge load at the worst point which is either between points 3 and 2 or between points II and III. Allowing for a " spread " at 60° the moment due to the knife-edge load is 2700 times the average ordinate of the area shown hatched in dotted lines on the moment influence line or 2700 × − 0·43 = − 1,160 lbs.-ft. The thrust due to the knife-edge load is likewise 2,700 times the average ordinate of the area shown hatched in dotted lines on the thrust influence line or 2,700 × 0·716 = 1930 lbs. The distributed load to be applied for negative moment is specified as 220 lbs. per sq. ft. causing a moment of 220 times the area of both negative parts of the moment influence line and a thrust of 220 times the areas *p q u* and *r s t* of the thrust influence line. These values are

$$-\,0{\cdot}0219 \times 220 \times 17{\cdot}5^2 = -\,1480 \text{ lbs.-ft. and}$$

$$0{\cdot}265 \times 220 \times \frac{17{\cdot}5^2}{6} = 2980 \text{ lbs.}$$

Collecting these results :—

Maximum positive live load moment at crown

$$\begin{aligned} &= 2700 \times (+\,1{\cdot}49) &&= +4030 \text{ lbs.-ft.} \\ &+\,295 \times 0{\cdot}73 \times 9{\cdot}17 &&= +1980 \text{ ,, ,,} \\ & &&\quad \overline{+6010 \text{ ,, ,,}} \end{aligned}$$

Corresponding thrust

$$\begin{array}{rcl} 2700 \times 1{\cdot}36 & = & 3670 \text{ lbs.} \\ + \quad 0{\cdot}235 \times 295 \times \dfrac{17{\cdot}5^2}{6} & = & 3500 \text{ ,,} \\ & & \overline{7220 \text{ ,,}} \end{array}$$

Maximum negative live load moment at crown

$$\begin{array}{rcl} = 2700 \times -0{\cdot}43 & = & -1160 \text{ lbs.-ft.} \\ -\ 0{\cdot}0219 \times 220 \times 17{\cdot}5^2 & = & -1480 \text{ ,, ,,} \\ & & \overline{-2640 \text{ ,, ,,}} \end{array}$$

Corresponding thrust

$$\begin{array}{rcl} 2700 \times 0{\cdot}716 & = & 1930 \text{ lbs.} \\ +\ 0{\cdot}265 \times 220 \times \dfrac{17{\cdot}5^2}{6} & = & 2980 \text{ ,,} \\ & & \overline{4910 \text{ ,,}} \end{array}$$

For the temperature, shrinkage and crown-drop stresses we must fix a value for the area at the crown and the value of I at the crown section. Assuming 1% of steel top and bottom we may take the area as

$$8'' \times 12'' + 14 \times \frac{8 \times 12}{100} \times 2$$
$$= 123 \text{ sq. ins. (concrete).}$$

We may take I as

$$\frac{12 \times 8^3}{12} + {\cdot}96 \times 3^2 \times 14 \times 2$$
$$= 512 + 242$$
$$= 754 \text{ ins.}^4 \text{ (concrete).}$$

If we take the coefficient of thermal expansion Φ as 0·000006 per 1° F. and $E_c = 2{,}000{,}000$ lbs. per sq. in., then a thrust causing a compression of 12 lbs. per sq. in. causes exactly as much shortening as a fall in temperature of 1° F. To cause a stress of 12 lbs. per sq. in. on the crown section we must have a thrust of $12 \times 123 = 1480$ lbs. Of course the section of the arch increases as we approach the springing, but we may neglect this small variation in the crown-drop. Any thrust H causes an effect similar to a fall in temperature

$$\frac{H}{1480} \text{ degrees F.}$$

A shrinkage of 0·02% causes a unit shortening of 0·0002 equal to a fall in temperature of 33·3° F. From Table A IV the thrust and moment at the crown section due to a fall of R° F. are

$$H = -\frac{45\ E\ I_e\ \Phi\ R}{4\ b^2}$$

$$M = +15\,\frac{E\,I_0\,\Phi\,R}{4b}$$

as $b = 72$ ins.

$$H = -\frac{45 \times 2{,}000{,}000 \times 754 \times 0{\cdot}000006\,R}{4 \times 72^2}$$

$$M_0 = +\frac{15 \times 2{,}000{,}000 \times 754 \times 0{\cdot}000006\,R}{4 \times 72}$$

$H \;= -\,19{\cdot}7\,R$ lbs.
$M_0 \;= +\,472\,R$ lbs.-ins. $= +\,39{\cdot}3\,R$ lbs.-ft.

A fall of 30° F. causes
$H \;= -\,19{\cdot}7 \times 30 = -\,592$ lbs.
$M_0 \;= +\,472 \times 30 = +\,14{,}100$ lbs.-ins. $= +\,1180$ lbs.-ft.

A shrinkage of 0·02% causes
$H \;= -\,19{\cdot}7 \times 33{\cdot}3 = -\,655$ lbs.
$M_0 \;= +\,472 \times 33{\cdot}3 = +\,15{,}700$ lbs.-ins. $= +\,1310$ lbs.-ft.

A thrust of 1480 lbs. causes a shortening equal to a fall of 1° F. and induces
$H \;= -\,19{\cdot}7$ lbs.
$M_0 \;= +\,472$ lbs.-ins. $= +\,39{\cdot}3$ lbs.-ft.

The greatest positive moment at the crown is obtained by applying the worst possible positive live loading with a fall in temperature. The greatest possible negative moment at the crown is obtained by applying the worst possible negative live loading with a rise in temperature. In both cases, of course, the dead load and shrinkage effects will always be present.

To obtain the greatest positive moment :—

	Moment.	Thrust.
Dead Load	− 1795 lbs.-ft.	7754 lbs.
Positive Live Load	+ 6010 „ „	7220 „
Temperature (Fall)	+ 1180 „ „	− 592 „
Shrinkage	+ 1310 „ „	− 655 „
	+ 6705 „ „	13,727 „
Crown-drop = $\left(\frac{13{,}727}{1480}\right)$ units =	+ 363 „ „	− 183 „
	+ 7068 „ „	13,544 „

To obtain the greatest negative moment :—

	Moment.	Thrust.
Dead Load	− 1795 lbs.-ft.	7754 lbs.
Negative Live Load . . .	− 2640 „ „	4910 „
Temperature (Rise) . . .	− 1410 „ „	+ 592 „
Shrinkage	+ 1570 „ „	− 655 „
	− 4275 „ „	12,601 „
Crown-drop = $\left(\frac{12{,}601}{1480}\right)$ units =	+ 335 „ „	− 169 „
	− 3940 „ „	12,432 „

As the shrinkage stresses reduce the negative moment and as we cannot absolutely rely on the shrinkage it might be safer to omit it from the negative moment table which would give a moment of — 5493 lbs.-ft. and a thrust of 13,079 lbs. In any case the conditions under positive moment are more severe and the section must resist a moment of 7068 lbs.-ft. with a thrust of 13,544 lbs.

Following Chapter VIII

$$M = 7068 \times 12 = 85{,}000 \text{ lbs.-ins.}$$

$$e = \frac{85{,}000}{13{,}544} = 6{\cdot}29 \text{ ins.}$$

$$\frac{L}{ab} = \frac{13{,}544}{8 \times 12} = 141$$

$$\frac{e}{a} = \frac{6{\cdot}29}{8} = 0{\cdot}785.$$

With $f_c = 750$, then $\frac{L}{ab} \times \frac{600}{f_c} = 113.$

From Table C III (*a*) we need p = about 1½% and the value of n is about 0·43

$$\text{If } f_c = 750 \text{ then } f_s = \frac{750 \times 15 \times {\cdot}47}{{\cdot}43} = 12{,}300 \text{ lbs. per sq. in.}$$

$$1\tfrac{1}{2}\% = \frac{1{\cdot}5 \times 96}{100} = 1{\cdot}44 \text{ sq. ins. or } 0{\cdot}72 \text{ sq. ins. top and bottom.}$$

Say ⅝-in. bars at 5-in. centres top and bottom.

If we could increase f_c to 900 lbs. per sq. in. we could reduce the steel to ⅝-in. every 8 ins. top and bottom.

This section has been set out at great length and although the other sections entail calculating the vertical shear, the values may all be tabulated and the working shortened.

In preliminary designs it is only necessary to examine the crown and springing sections. In final designs at least one further section should be checked.

Spans up to 50 ft. 0 ins.—Conclusion.—The vexed questions of maximum live load and impact are generally now fixed by some authority. From the example we see that a parabola with filled-in spandrels has negative moment at the crown and at the springing under dead load alone. The crown-drop and shrinkage stresses produce positive moment at the crown and negative moment at the springing. At the crown section the maximum positive moment is generally greater than the maximum negative moment. At the springing section the maximum negative moment is generally much larger than the maximum positive moment. This high value of the negative moment at the springing section is very local and falls off very rapidly. The parabola is, from a general standpoint, the best shape for small arches. If from questions of headroom or appearance a semi-ellipse (or five-centre) arch is decided on the values in Tables A III, A V and VI may be used. The semi-elliptical arch may have to be an inch or two thicker at the crown than a parabolic arch of similar span and rise.

As the arch ring is under compression a high value of the safe unit shear may be taken. A small arch would have no shear reinforcement but would nevertheless be capable of taking about 200 lbs. per sq. in. on the *b j d* area. In the example shown in Fig. 173 the true shear on the springing section (which is merely that resultant of the total thrust which acts transversely to the section) would never exceed about 5000 lbs. As the section is 12 ins. thick the *b j d* area is about 116 sq. ins. and would safely take a true shear of about 23,200 lbs.

There may be some small horizontal pressure on the arch from the spandrel filling, but as this is always selected material or hardcore the effect is negligible.

Arrangement of Bars.—The main bars in small arches are placed in two layers one at the top and one at the bottom. No bent-up bars are employed and the main bars are laid in as long lengths as possible and lapped 40 to 45 diameters. Care must be taken to distribute the overlaps as evenly as possible. Cross bars are placed under the top layer and over the bottom layer. The area of these cross bars may be about one-quarter that of the main bars. In the example in Fig. 173 say $\frac{1}{2}$-in. bars at 12-in. centres top and bottom. To hold the top bars in place during concreting one $\frac{3}{8}$-in. stirrup per sq. yd. of arch may be provided.

Spans exceeding 50 ft. 0 ins.—To avoid the weight of filling necessary to fill in the spandrels, large arches are built in the form of arched ribs with a roadway supported on columns. As an alternative the arch may be built as a sheet supporting reinforced concrete cross walls which support the roadway, the spandrels being either closed in or left open according to taste. In all such cases where the load is applied at definite points the line of thrust will consist of a series of straight lines. The arch itself is always made in the shape of a continuous curve. For large arched ribs the yielding of the abutments will probably be less in proportion to the span and the conditions of loading will be more definite. On the other hand discontinuous concreting and deflection of the falsework may introduce considerable initial stresses.

It is very doubtful in our present state of knowledge (or rather ignorance) of the elastic properties of practical concrete whether we are justified in applying exact mathematical methods to arch design. It is, however, fashionable to assume such treatment of the subject is justified. It is possible that we may, at some future date, be able to produce uniformly strong and uniformly elastic concrete, but such things are not yet in sight. On the other hand it is only fair that the designer should be able to find out what happens when he alters the shape or thicknesses in his design, provided always that he keeps a proper balance in his mind between theory and practice. For a complete treatment of larger spans of all types and discussions on movements of abutments or elastic piers the reader is referred to the author's *Reinforced Concrete Arch Design.*

The Semicircular Encastré Arch.—(**$E\,I$ constant**).—This case sometimes arises in roof work and the results may be useful in dealing with other circular structures such as large round culverts, large silos under wind load, etc. (The analysis was run out by the author when designing some lattice ribs for a steel roof.)

As only the special case of a whole semicircle will be considered we shall perform the integration across the whole arch at once.

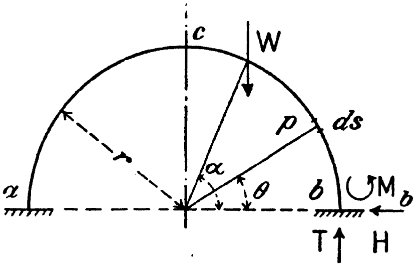


Fig. 174.

Let Fig. 174 represent a semicircular encastré arch whose radius is r.

Take the origin at a and $a\,b$ as axis of x. At any point p we can write everything in terms of r and the angle θ

$$x = r\,(1 + \cos\theta)$$
$$y = r\sin\theta$$
$$d\,s = r\,d\,\theta$$

Let M_b, H and T be the reactions at the right hand abutment. Imagine a vertical load W applied at a point subtending an angle α.

Let M = the bending moment at any point then

$$M = M_b + T\,(r - r\cos\theta) - H\,r\sin\theta$$

until we reach a value $\theta = \alpha$ when an additional term enters namely

$$-\ W\,(r\cos\alpha\ -\ r\cos\theta)$$

In integrating the slopes and deflections we must therefore integrate the first three terms from o to π and the last term only from α to π.

We have $\displaystyle\int \frac{M}{E\,I}d\,s = 0$ over the whole arch

$\therefore$ $\displaystyle\int M\,d\,s = 0$ since $E\,I$ = a constant i.e., $$0 = \int_o^{\pi} (M_b - H\,r \sin\theta + T\,(r - r\cos\theta)\,)\,r\,d\,\theta$$

$$- \int_a^{\pi} W\,(r\cos a - r\cos\theta)\,r\,d\,\theta$$

$$0 = (M_b\pi - 2\,H\,r + \pi\,T\,r) - W\,r\,(\cos a\,(\pi - a) + \sin a) \quad . \; . \; . \quad (1)$$

$$\int_o^{\pi} \frac{M}{E\,I}\,x\,d\,s = \int_o^{\pi} \frac{M}{E\,I}\,(r + r\cos\theta)\,d\,s = 0$$

$$= r\int_o^{\pi} \frac{M}{E\,I}\,d\,s + r\int_o^{\pi} \frac{M}{E\,I}\cos\theta\,d\,s$$

Now by our previous condition $r\int_o^{\pi} \frac{M}{E\,I}\,d\,s = 0$

$$\therefore \qquad \int_o^{\pi} \frac{M}{E\,I}\cos\theta\,d\,s = 0 = \int_o^{\pi} M\cos\theta\,d\,s$$

$$0 = \int_o^{\pi} (M_b\cos\theta - H\,r\sin\theta\cos\theta + T\,r\cos\theta - T\,r\cos^2\theta)\,r\,d\,\theta$$

$$- \int_a^{\pi} W\,r\,(\cos a\cos\theta - \cos^2\theta)\,r\,d\,\theta$$

$$\text{i.e.,}\quad T = \frac{W}{\pi}\,(\pi - a + \cos a\sin a) \quad . \; . \; . \; . \quad (2)$$

Also $$\int \frac{My}{E\,I}\,d\,s = 0 = \int M\,y\,d\,s = \int M\,r\sin\theta\,r\,d\,\theta$$

$$\therefore \qquad \int M\sin\theta\,d\,\theta = 0$$

$$0 = \int_o^{\pi} (M_b\sin\theta - H\,r\sin^2\theta + T\,r\sin\theta - T\,r\sin\theta\cos\theta)\,d\,\theta$$

$$- W\,r\int_a^{\pi} (\cos a\sin\theta - \cos\theta\sin\theta)\,d\,\theta$$

$$= \left(2M_b - \frac{\pi}{2}H\,r + 2\,T\,r\right) - W\,r\left(\cos a + \cos^2 a + \frac{\sin^2 a}{2}\right) \quad . \quad (3)$$

Equation (2) gives T directly. By substituting this value in equations (1) and (3) and then eliminating M_b and H separately we arrive at values for H and M_b as follows.

$$H = \frac{4\,\pi}{\pi^2 - 8} \times W\left(\frac{\cos a}{2} - \frac{a\,(\cos a)}{\pi} + \frac{\sin a}{\pi} - \frac{\sin^2 a}{4} - \frac{\cos^2 a}{2}\right) . \; (4)$$

$$M_b = \frac{W r}{2}\Big[6{\cdot}25\cos\alpha - 2{\cdot}13\cos^2\alpha - 3{\cdot}35\,\alpha\cos\alpha + 3{\cdot}33\sin\alpha$$

$$+ {\cdot}638\,\alpha - {\cdot}638\cos\alpha\sin\alpha - 4{\cdot}12\Big] \quad . \; . \; . \; . \; . \quad (5)$$

As semicircular ribs may be used in roof work where horizontal loads occur we must also examine the stresses under a horizontal point load P. In Fig. 175 we have the case as shown in Fig. 174 but having a horizontal load substituted for the vertical one. We see that the last term in our integration becomes

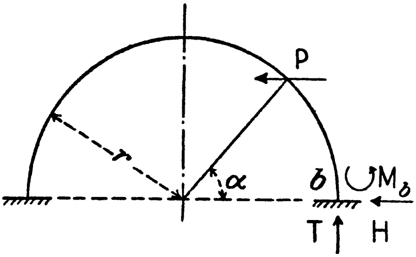


Fig. 175.

$$- P r(\sin\theta - \sin\alpha) \text{ instead of } - W(r\cos\alpha - r\cos\theta).$$

The first three terms will remain as before.

This gives us

$$0 = (M_b\pi - 2Hr + \pi T r) - P r(1 + \cos\alpha + \alpha\sin\alpha - \pi\sin\alpha) \quad (6)$$

$$T = -\frac{P\sin^2\alpha}{\pi} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

$$0 = \left(2M_b - \frac{\pi}{2}Hr + 2Tr\right) - Pr\left(\frac{\pi-\alpha}{2} - \frac{\sin\alpha\cos\alpha}{2} - \sin\alpha\right). \quad (8)$$

$$\text{giving } H = \frac{2}{(\pi^2 - 8)} \times P\Big(2 + 2\cos\alpha + 2\alpha\sin\alpha - \pi\frac{(\pi-\alpha)}{2}$$

$$+ \frac{\pi\sin\alpha\cos\alpha}{2} - \pi\sin\alpha\Big) \quad . \; . \; . \; . \; . \; . \quad (9)$$

$$\text{and } M_b = \frac{Pr}{2}(2{\cdot}13\alpha - 3{\cdot}34 - 6{\cdot}25\sin\alpha + 2{\cdot}12\sin\alpha\cos\alpha$$

$$+ {\cdot}637\sin^2\alpha + 3{\cdot}34\cos\alpha + 3{\cdot}34\,\alpha\sin\alpha) \quad . \; . \; . \quad (10)$$

In equations (5) and (10) approximate values of the coefficients have been inserted. The results of these equations are shown in Table A VII. The values of M_b have been obtained by slide-rule from equations (5) and (10) and are therefore not correct to ± 5%.

A rise of temperature of $R°$ gives

$$H = \frac{6{\cdot}74\,\phi R E I}{r^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

$$M_b = \frac{4{\cdot}28\,\phi\,R\,E\,I}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Where the symbols have the values in Table A IV.

TABLE A VII

SEMI-CIRCULAR ENCASTRÉ ARCH *EI* = constant							
Vertical point load as Fig. 174				Horizontal point load as Fig. 175.			
a°	M_b	*H*	*T*	*a*°	M_b	*H*	*T*
0	0	0	1·00 *W*	0	0	− 1·00 *P*	0
10	− ·015 *Wr*	·010 *W*	·998 *W*	10	− ·148 *Pr*	− ·945 *P*	− ·010 *P*
20	− ·031 *Wr*	·035 *W*	·993 *W*	20	− ·228 *Pr*	− ·880 *P*	− ·037 *P*
30	− ·050 *Wr*	·067 *W*	·970 *W*	30	− ·256 *Pr*	− ·792 *P*	− ·080 *P*
40	− ·045 *Wr*	·140 *W*	·935 *W*	40	− ·252 *Pr*	− ·678 *P*	− ·131 *P*
50	− ·035 *Wr*	·214 *W*	·882 *W*	50	− ·239 *Pr*	− ·612 *P*	− ·188 *P*
60	+ ·003 *Wr*	·307 *W*	·806 *W*	60	− ·219 *Pr*	− ·572 *P*	− ·240 *P*
70	+ ·041 *Wr*	·387 *W*	·714 *W*	70	− ·202 *Pr*	− ·525 *P*	− ·282 *P*
80	+ ·085 *Wr*	·430 *W*	·612 *W*	80	− ·195 *Pr*	− ·510 *P*	− ·308 *P*
90	+ ·110 *Wr*	·454 *W*	·500 *W*	90	− ·183 *Pr*	− ·500 *P*	− ·318 *P*
100	+ ·130 *Wr*	·430 *W*	·388 *W*	100	− ·180 *Pr*	− ·490 *P*	− ·308 *P*
110	+ ·123 *Wr*	·387 *W*	·286 *W*	110	− ·174 *Pr*	− ·475 *P*	− ·282 *P*
120	+ ·110 *Wr*	·307 *W*	·194 *W*	120	− ·167 *Pr*	− ·428 *P*	− ·240 *P*
130	+ ·086 *Wr*	·214 *W*	·118 *W*	130	− ·151 *Pr*	− ·388 *P*	− ·188 *P*
140	+ ·059 *Wr*	·140 *W*	·065 *W*	140	− ·129 *Pr*	− ·322 *P*	− ·131 *P*
150	+ ·024 *Wr*	·067 *W*	·030 *W*	150	− ·084 *Pr*	− ·208 *P*	− ·080 *P*
160	+ ·010 *Wr*	·035 *W*	·007 *W*	160	− ·040 *Pr*	− ·120 *P*	− ·037 *P*
170	+ ·001 *Wr*	·010 *W*	·002 *W*	170	− ·006 *Pr*	− ·055 *P*	− ·010 *P*
180	0	0	0	180	0	0	0
Positive Moments have tension on intrados.							

The integration of *H* and M_b for distributed loads is rather long and such loads are best split up into a series of point loads.

Extension to Circular Pipes.—In cases such as large circular culverts under a deep fill, etc., we may arrive at the bending moments by referring to our equations of deflection and choosing those that apply to the special case in point. For example if we have a pipe

10 ft. 0 ins. in diameter buried under 50 ft. 0 ins. of fill which has an instantaneous angle of repose of 30° we may take this as exerting a vertical pressure of $\frac{w\,r\cot\theta}{2}$ per sq. ft. and a horizontal pressure of $\frac{w\,r\cot\theta}{2}\times\frac{1}{3}$ per square foot where $\theta = 30°$. Putting $w = 120$ lbs. cub. ft. we have vertical pressure

$$= 5 \times 104 \text{ lbs. per sq. ft.}$$
$$= 520 \text{ lbs. per sq. ft.}$$

horizontal pressure $= 173$,, ,, ,,

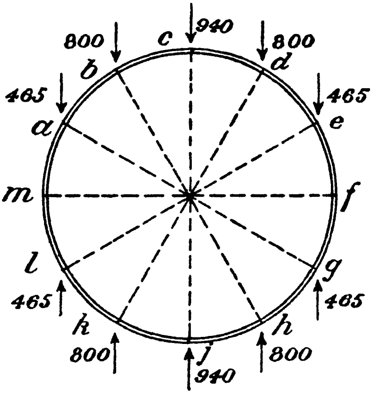


Fig. 176.

These two pressures can be replaced by a radial pressure of 173 lbs. per sq. ft. and a vertical pressure of

$$520 - 173 = 347 \text{ lbs. per sq. ft.}$$

Total vertical pressure

$$= 347 \times 10 = 3470 \text{ lbs. per foot run of pipe.}$$

The radial pressure produces a compression of

$$5 \times 173 \text{ lbs.} = 865 \text{ lbs. per foot run of pipe}$$

acting as a direct pressure at all sections.

The vertical pressure may be split up into 5 point loads top and bottom the pipe being divided into 12 sectors of 30° each. Say 940 lbs. at the top and bottom points, 800 and 465 lbs. at the other load points. If we consider the top half of the pipe we see that the crown tends to flatten and the sides tend to move out. Theoretically speaking the tendency of the side to move outward against the earth should produce a counter pressure inwards. Looked at from the practical standpoint this counter pressure could only be relied on if the filling were perfectly consolidated round the pipe. It is well known that all filling, however well rammed, shrinks after being placed. We cannot therefore rely on counter pressure in any practical case unless the structure is free to move and follow the earth as it shrinks and compresses. In fact it is doubtful if we are justified in relying on the active earth pressure in any case where such pressure tends to diminish the stresses. We shall therefore assume that the pipe itself resists all the flattening action. Taking the diameter *m f* in Fig. 176 we see that by symmetry there can be no horizontal reaction of the top half of the pipe on the bottom half and vice versa. The vertical reaction, by symmetry, is $\frac{3470}{2} = 1735$ lbs. at each side.

As we assume that the pipe flattens out under load we see that the distance *m f* does not remain constant ∴ equation (3) does not hold. Both *m* and *f* remain at the same level and the slopes at *m* and *f* remain vertical ∴ equations (1) and (2) will hold good. Equation (2) gives the vertical reactions which we know already by symmetry. We have only to deal with equation (1) (a = angle in radians)

$$M_t \pi - 2 H r + \pi T r = W r (\cos a (\pi - a) + \sin a) \quad . \quad . \quad . \quad (1)$$

We have already $H = 0$

Total $T = 1735$ lbs. (due to all five loads)

$r = 5$ feet

$$\therefore \text{Total } M_t = -1735 \times 5 + \Sigma \frac{W r}{\pi} (\cos a (\pi - a) + \sin a).$$

Taking the loads separately :—

465 lbs. at $a = 30°$ gives

$$\frac{W r}{\pi} (\cos a (\pi - a) + \sin a) = \frac{465 \times 5}{3{\cdot}14} \left({\cdot}866 \left(\frac{5 \times 3{\cdot}14}{6} \right) + {\cdot}5 \right)$$

800 lbs. at $a = 60°$ gives $\frac{800 \times 5}{3{\cdot}14} \left(\frac{1}{2} \left(\frac{3{\cdot}14 \times 2}{3} \right) + {\cdot}866 \right)$

940 lbs. at $a = 90°$ gives $\frac{940 \times 5}{3{\cdot}14} (1)$

800 lbs. at $a = 120°$ gives $\frac{800 \times 5}{3{\cdot}14} \left(-\frac{1}{2} \left(\frac{3{\cdot}14}{3} \right) + {\cdot}866 \right)$

465 lbs. at $a = 150°$ gives $\frac{465 \times 5}{3{\cdot}14} \left(- {\cdot}866 \times \left(\frac{3{\cdot}14}{6} \right) + {\cdot}5 \right)$

$$\Sigma \frac{W r}{\pi} (\cos a (\pi - a) + \sin a) = 1500 + 2870 + 2090 = 6460.$$

Total M_t due to all five loads

$$= -8650 + 6460$$
$$= -2190 \text{ lbs.-ft.}$$

That is to say the bending moment at f (and also by symmetry at m) which corresponds to M_b in the equation (compare Fig. 174 with Fig. 176) is − 2190 lbs.-ft. per foot run of pipe.

At this section we have also a reaction T of 1735 lbs. and a pressure caused by the outside radial pressure of 865 lbs. or a total direct compression of 2600 lbs. with a bending moment of 26,200 lbs.-in. giving an eccentricity of 10 ins. By statics the bending moment at the crown is + 2440 lbs.-ft. with a direct pressure, due to the radial pressure, of 865 lbs.

The reader must distinguish carefully between the case just worked out where the diameter $m\ f$ increases under load and the case tabulated in Table A VII which represents an arch supported on unyielding abutments. Most cases of culverts are symmetrical about some axis which makes it possible to write down some of the stresses by inspection and also to use one or more of the equations for the semicircular arch without bothering to integrate them separately for each case.

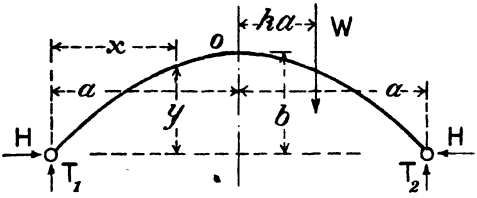


Fig. 177.

Two-Hinged Arches. Parabolic Arch.—As a rule, if hinges are introduced into an arch, then three are provided. Pure two-hinged arches are not common. In some cases it is not possible to fix the ends of an arch satisfactorily and the structure, although not actually provided with hinges, becomes more or less a two-hinged arch.

Let Fig. 177 represent a two-hinged parabolic arch of 2 a span and b rise carrying a load W. By taking moments about one hinge and then about the other we have

$$T_2 \times 2\,a = W\,(1 + k)\,a$$
$$T_1 \times 2\,a = W\,(1 - k)\,a$$

The total slope measured over the whole arch does not necessarily remain constant as the slopes are not fixed at the hinges. If we attempt to use the fact that the two hinges remain at the same level we are forced to introduce values of the slopes at certain points as a change of slope at one hinge changes the origin from which we measure our vertical deflections (compare the deflections in continuous beams as drawn and calculated in Chapter X). This prevents our taking the origin for x and y at the crown. We must therefore measure x and y as shown in Fig. 177. In order to get a definite numerical result we shall assume as we did in the case of the encastré arch that $\dfrac{d\,s}{I\,d\,x}$ = constant.

The equation of the parabola referred to the crown as origin is

$$y = -\frac{b}{a^2}x^2.$$

Taking the origin to one hinge

$$y = b - \frac{b}{a^2}(a - x)^2$$

$$y = \frac{b}{a^2}[2\,a\,x - x^2].$$

As we have the analysis for the encastré arch in full the analysis for the two-hinged type will be omitted.

The result is

$$H = \frac{5}{16}W\left[2\,(1+k) - (1+k)^3 + \frac{(1+k)^4}{4}\right]\frac{a}{b}\,\cdot$$

When we know the value of H we have a full solution of the problem.

For $k = 0$, $H = \cdot 392\,W\frac{a}{b}$ and $M_o = + \cdot 108\,W\,a$

$k = \pm \cdot 2$, $H = \cdot 372\,W\frac{a}{b}$ and $M_o = + \cdot 028\,W\,a$

$k = \pm \cdot 4$, $H = \cdot 3175\,W\frac{a}{b}$ and $M_o = - \cdot 0175\,W\,a$

$k = \pm \cdot 6$, $H = \cdot 232\,W\frac{a}{b}$ and $M_o = - \cdot 032\,W\,a$

$k = \pm \cdot 8$, $H = \cdot 1226\,W\frac{a}{b}$ and $M_o = - \cdot 0226\,W\,a.$

The reader may easily draw out the influence lines for himself should he ever require them.

Temperature Stresses.—The horizontal expansion of the arch under temperature $= 2\,a\,\Phi\,R$ using the same notation as in Table A IV.

The horizontal deflection of the arch which we have already calculated is

$$\int \frac{M}{E\,I}\,y\,d\,s.$$

Since W is zero and T_1 and T_2 vanish

$$M = -\,H y$$

$$\int \frac{M}{E\,I}\,y\,d\,s = -\,\frac{1}{E\,I_o}\int_0^{2a} H y^2\,d\,x$$

$$= -\,\frac{1}{E\,I_o}\int_0^{2a} \frac{b^2}{4\,a^4}[4\,a^2\,x^2 - 4\,a\,x^3 + x^4]\,H\,d\,x.$$

The horizontal expansion due to temperature must equal the horizontal deflection due to the value of H which is caused by the temperature change.

$$\therefore \pm \frac{16}{15} H \frac{b^2}{a^4} \times \frac{1}{E I_o} a^5 = 2 a \Phi R$$

$$H = \pm \frac{30}{16} \frac{\Phi R E I_o}{b^2}$$

If E = 2,000,000 lbs. per sq. in. and $\Phi = \cdot 000006$

$$H = \pm \frac{22{\cdot}5 \, I_o R^\circ}{b^2}$$

where I_o is in inch units (concrete units) and b is in inches. The resulting value of H is in lbs.

Notice how much smaller this result is than the corresponding thrust in an encastré arch due to the fact that the bending moment has the same sign everywhere. In an encastré arch the reversed moments at the abutments cause elongation when the moment at the crown produces shortening and vice versa, requiring a much larger thrust to cause the same net result. The largest bending moment due to temperature occurs at the crown and is :—

$$M_o = \mp \frac{22{\cdot}5 \, I_o R^\circ}{b}.$$

Two-Hinged Semicircular Arch.—*E I* constant.

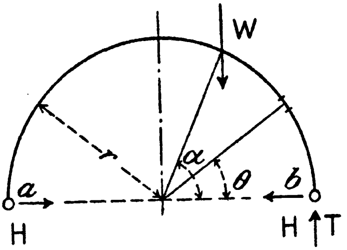


Fig. 178.

Let Fig. 178 represent a semicircular arch hinged at a and at b. We shall use the same notation as in Fig. 174 and apply the condition

$$\int \frac{M}{E I} y \, d s = 0.$$

The two-hinged arch may be looked upon as a special case of the encastré arch

where $M_b = 0$

and $T = \dfrac{W(1 + \cos a)}{2}.$

Using equation (3) from the encastré arch in Fig. 174 and putting in the two values above, we have by integration :—

$$H = \frac{W \sin^2 a}{\pi}$$

Similarly taking the same arch under a horizontal point load P as in Fig. 179, the value of H in the direction shown is

$$H = \frac{P}{\pi}\left((\pi - a) - \sin\frac{2a}{2}\right)$$

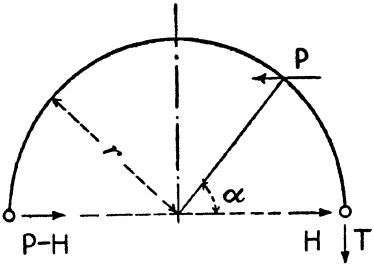


Fig. 179.

A rise of temperature causes a positive horizontal thrust through each hinge of

$$\frac{4\,E\,I\,\Phi\,R}{\pi\,r^2}$$

where the symbols have the meanings in Table A IV.

The results are collected in Table A VIII.

TABLE A VIII

SEMI-CIRCULAR TWO-HINGED ARCH EI = constant

Vertical Point Load as Fig. 178		Horizontal Point Load as Fig. 179.	
$a°$	H	$a°$	H
0	0 ·	0	1·000 P
10	·009 W	10	·890 P
20	·037 W	20	·788 P
30	·080 W	30	·695 P
40	·132 W	40	·620 P
50	·188 W	50	·565 P
60	·240 W	60	·530 P
70	·282 W	70	·510 P
80	·310 W	80	·500 P
90	·317 W	90	·500 P

Temperature Stresses, $H = \pm \frac{4\,E\,I\,\Phi\,R^{\circ}}{\pi\,r^2}$

Arches—Conclusion.—Graphical methods of analysis are apt to prove long and misleading but at least two lines of thrust should be drawn out from the calculated values, one for the crown and one for the abutment section. These will prove a useful check and may reveal high stresses where not anticipated. If the foundations are not good enough to carry abutments for an arch, it may be possible to use a continuous double cantilever which will have the appearance of an arch and which requires no lateral support from the abutments.

Two-Hinged Tied Arches.—For a discussion of these see the author's *Reinforced Concrete Arch Design.*

Three-Hinged Arches.—These structures are merely bent members. The moments and thrusts are found by statics.

Circular Pipes under Uniform Loads.—A thin circular pipe as in Fig. 179a subject to a uniformly distributed vertical load of w and a uniformly distributed horizontal pressure p has the following moments :—

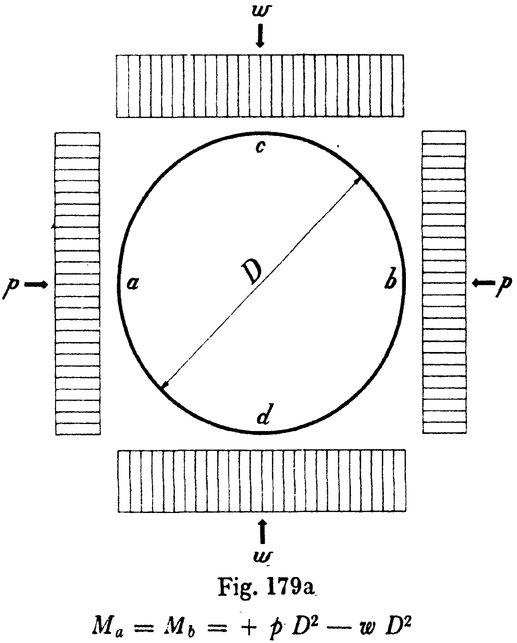


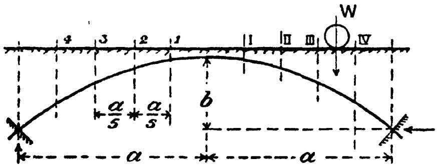
Fig. 179a

$$M_a = M_b = + \frac{p\,D^2}{16} - \frac{w\,D^2}{16}$$

$$M_c = M_d = + \frac{w\,D^2}{16} - \frac{p\,D^2}{16}$$

These figures are exact but apply only to uniformly distributed loads. Odd loading may be treated as in Fig. 176 by splitting into a series of point loads.

TABLE A I



ENCASTRÉ PARABOLIC ARCH $\frac{ds}{I\,dx}$ CONSTANT

$2a$ = SPAN
b = RISE
W = POINT LOAD

LOAD W AT	B.M. AT						HORIZONTAL THRUST	REACTION AT ABUTMENT
	ABUTMENT	4	3	2	1	CROWN		
4	− 0·1215	+ 0·051	+ 0·028	+ 0·011	− 0·002	− 0·010	0·061	0·972
3	− 0·1280	− 0·018	+ 0·107	− 0·048	+ 0·004	− 0·024	0·192	0·896
2	− 0·0735	− 0·036	+ 0·028	+ 0·119	+ 0·036	− 0·020	0·331	0·784
1	0	− 0·026	− 0·017	+ 0·026	+ 0·104	+ 0·016	0·432	0·648
CROWN	+ 0·0625	− 0·006	− 0·038	− 0·031	+ 0·020	+ 0·094	0·469	0·500
I	+ 0·0960	+ 0·011	− 0·040	− 0·056	− 0·037	+ 0·016	0·432	0·352
II	+ 0·0945	+ 0·018	− 0·031	− 0·054	− 0·050	− 0·020	0·331	0·216
III	+ 0·0640	+ 0·016	− 0·017	− 0·035	− 0·037	− 0·024	0·192	0·104
IV	+ 0·0225	+ 0·006	− 0·005	− 0·013	− 0·015	− 0·010	0·061	0·028
	× $W a$						× $W \frac{a}{b}$	× W

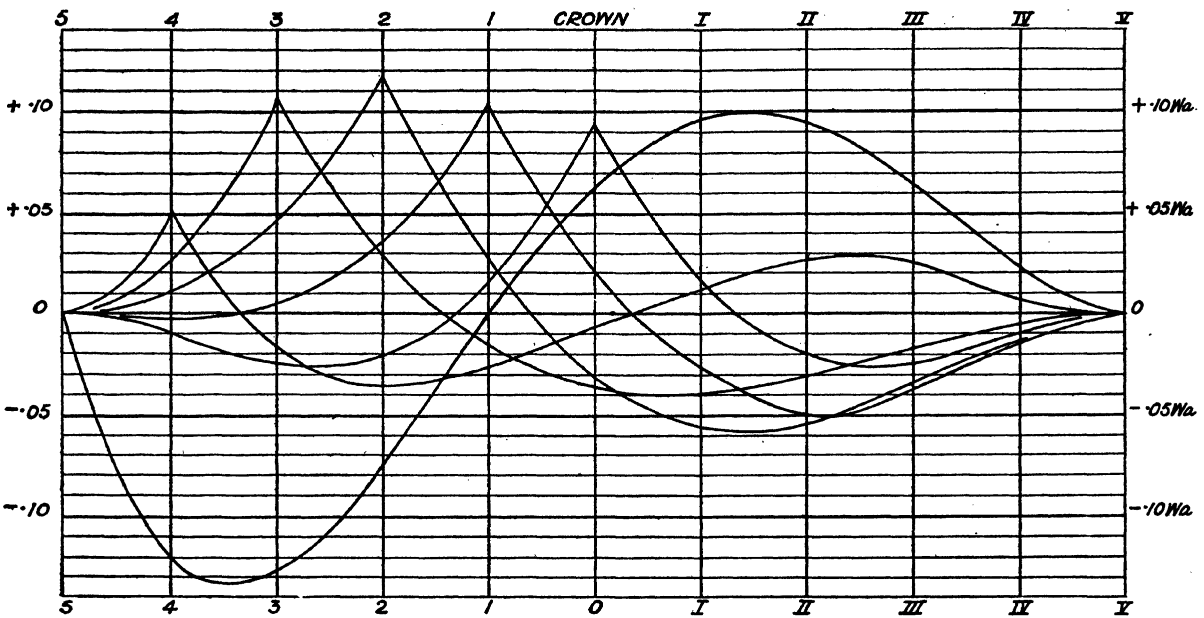
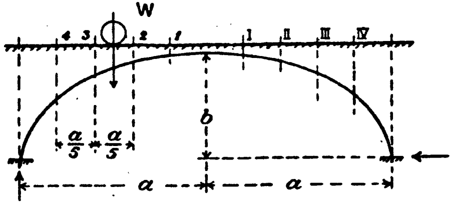


TABLE A II. BENDING MOMENT INFLUENCE LINES FOR ENCASTRÉ PARABOLIC ARCH SPAN 2a UNDER A POINT LOAD W.

$\frac{ds}{I\,dx}$ = CONSTANT

TABLE A III

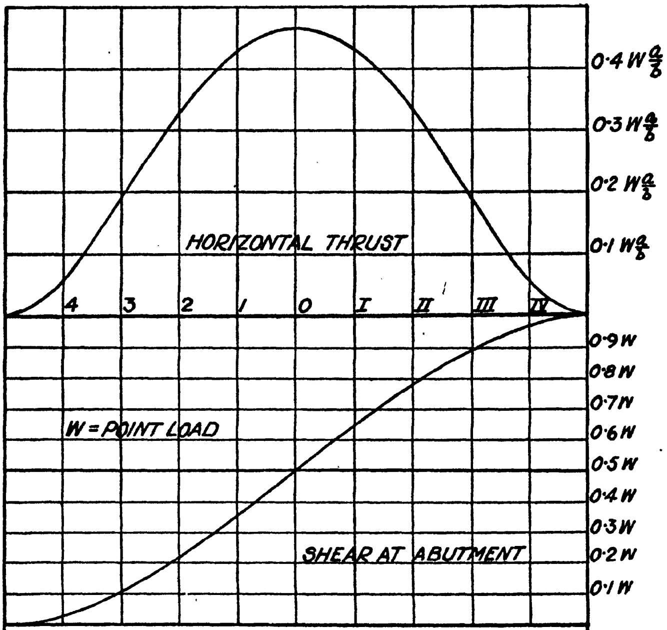


ENCASTRÉ SEMI-ELLIPTICAL ARCH $\frac{ds}{I\,dx}$ CONSTANT

(USE FOR 5 CENTRE)

$2a$ = SPAN
b = RISE
W = POINT LOAD

	BENDING MOMENT AT						HORIZONTAL THRUST	REACTION AT ABUTMENT
	5	4	3	2	1	0		
W ON POINT 4	− ·0897	+ ·0495	+ ·0255	+ ·0092	− ·0023	− ·0097	·0920	·972
,, ,, ,, 3	− ·0468	− ·0274	+ ·0985	+ ·0466	+ ·0089	− ·0172	·2664	·896
,, ,, ,, 2	+ ·0477	− ·0566	+ ·0133	+ ·1194	+ ·0486	− ·0033	·4349	·784
,, ,, ,, 1	+ ·1471	− ·0557	− ·0369	+ ·0282	+ ·1220	+ ·0411	·5541	·648
,, ,, ,, 0	+ ·2185	− ·0395	− ·0587	− ·0282	+ ·0339	+ ·1220	·5965	·500
,, ,, ,, I	+ ·2431	− ·0188	− ·0593	− ·0534	− ·0182	+ ·0411	·5541	·352
,, ,, ,, II	+ ·2157	− ·0022	− ·0460	− ·0534	− ·0372	− ·0033	·4349	·216
,, ,, ,, III	+ ·1453	+ ·0062	− ·0263	− ·0366	− ·0327	− ·0172	·2664	·104
,, ,, ,, IV	+ ·0543	+ ·0047	− ·0082	− ·0132	− ·0135	− ·0097	·0920	·028
	× *W a*						× $\frac{Wa}{b}$	× *W*



TEMPERATURE STRESSES $\left(\frac{\text{SUMMER}}{\text{WINTER}}\right)$

BENDING MOMENTS		THRUST
AT 5 ± 90·0	$\times \frac{I_o R^\circ}{b}$	$\pm \frac{135 I_o R^\circ}{b^2}$
AT 4 ± 41·8		
AT 3 ± 3·6		
AT 2 ∓ 39·6		
AT 0 ∓ 45·0		

NOTE a = HALF SPAN IN INCHES

b = RISE IN INCHES

I_o = MOMENT OF INERTIA AT CROWN INs.^{4} IN CONCRETE UNITS

R° = RISE OR FALL IN DEGREES FAHRENHEIT

E = 2,000,000 LBs. PER SQ. IN.

BENDING MOMENTS IN LBs. INs.

HORIZONTAL THRUST IN LBs.

TABLE A IV PARABOLIC ENCASTRÉ ARCH

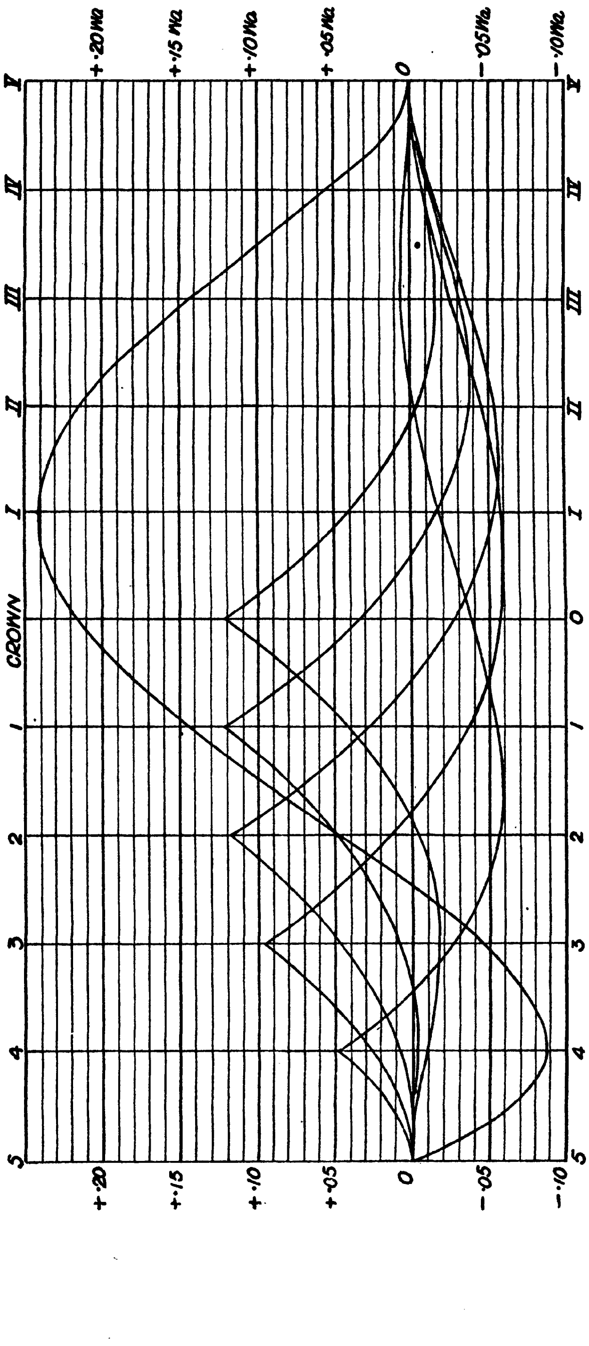
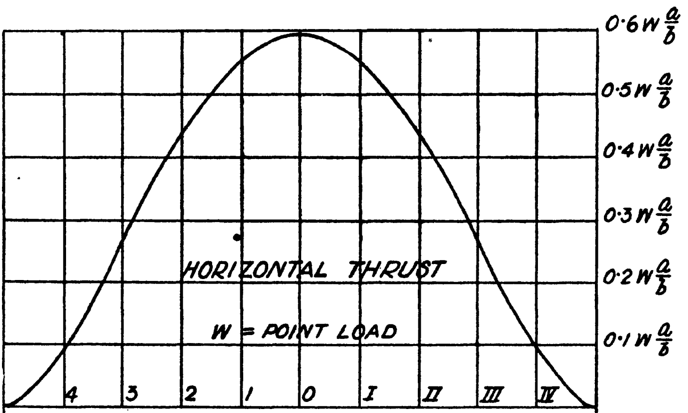


TABLE A V BENDING MOMENT INFLUENCE LINES FOR SEMI-ELLIPTICAL ENCASTRÉ ARCH SPAN 2a UNDER A POINT LOAD W.

$\frac{ds}{Idx}$ = CONSTANT



TEMPERATURE STRESSES $\left(\frac{\text{SUMMER}}{\text{WINTER}}\right)$

BENDING MOMENTS		THRUST
AT 5 ± 15·65	$\times \frac{E I_0 \phi R^\circ}{b}$	$\pm 20{\cdot}06 \frac{E I_0 \phi R^\circ}{b^2}$
AT 4 ± 3·719		
AT 3 ∓ ·293		
AT 2 ∓ 2·630		
AT 1 ∓ 3·904		
AT 0 ∓ 4·305		

NOTE

a = HALF SPAN IN INCHES

b = RISE IN INCHES

E = 2,000,000 LBs PER SQ. IN.

I_0 = MOMENT OF INERTIA AT CROWN INs.⁴ IN CONCRETE UNITS

ϕ = COEFFICIENT OF EXPANSION = ·000006

R° = RISE OR FALL IN DEGREES FAHRENHEIT

(TOTAL RANGE = $2R^\circ$)

BENDING MOMENTS IN LBs. INs.

HORIZONTAL THRUST IN LBs.

REACTION AT ABUTMENT AS IN TABLE A IV.

TABLE A VI. SEMI-ELLIPSE ENCASTRÉ ARCH.

$$\frac{ds}{I\,dx} = \text{CONSTANT.}$$

CHAPTER XIX

SIMPLE FRAMES

In this chapter each member will be assumed to be of constant cross section throughout its length and only straight members will be treated. The methods may be extended to members whose section varies or to curved members by introducing special factors, but so many combinations are possible that a complete book would be necessary to catalogue them all. (A particular case is given in Chapter XXXII.)

Moment of Inertia.—In general we require only the comparative values for each member and in most cases we may therefore disregard the steel reinforcement in all members and assume the section consists of solid uncracked concrete.

Elastic Modulus.—Despite the fact that the value of E_c under working loads may vary from below 1,500,000 to over 5,000,000 lbs. per sq. in. from one part of the structure to another it is usual to assume that E_c is a constant throughout.

Stiffness Factor.—In all frame questions the factor

$$\left(\frac{\text{moment of inertia}}{\text{length of member}}\right)$$

is constantly recurring and we may follow the D.S.I.R. code of 1934 and call this value K.

Movements of Joints.—When the floors are not uniformly loaded some columns may be fully loaded and some lightly loaded. The fully loaded columns will shorten more than the lightly loaded ones. Our theory of continuous beams assumes that all points of support remain at one level and this uneven sinking of the supporting columns introduces all sorts of unexpected moments. Any frame under eccentric loading tends to sway sideways and this adds more secondary stresses. In addition we may have unequal settlement of the footings and we always have shrinkage and temperature movements. It is quite beyond the limits of practical drawing-office methods to cope with all these factors in a complex frame and it is usual to neglect all vertical movements due to all causes. Side sway also is neglected except in very simple frames. This is generally justified as cross walls or end members prevent lateral movement in most large frames.

Shrinkage and temperature movements are seldom calculated, although most competent designers keep them in mind and provide

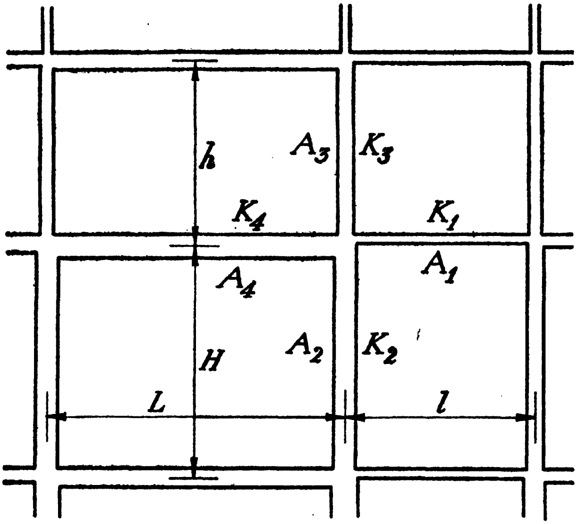


Fig. 180.

against them in a crude general way. This means that in most frames the only movements taken into account are the rotational movements of the joints.

Slope Deflection Method.—Chapter X furnishes an example of this method. In some cases it will be found to give the quickest and simplest solution. In other cases an application of Maxwell's

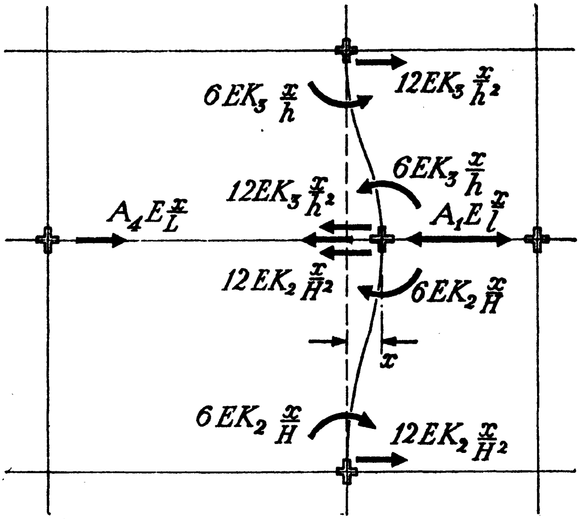


Fig. 181.

theorem will save time and labour, but in most frame problems a displacement method gives an easier solution.

Displacement Method.—If we know the moments, thrusts and shears at all points we can, by applying a mathematical process, calculate the rotations and movements of all joints in the frame. It is equally true that, if we know the rotations and movements of all the joints in the frame, we can, by applying a reverse mathematical process, calculate the moments, thrusts and shears. The method consists of two parts. First, all joints are assumed to remain absolutely fixed in position and direction and the moments due to the applied loads are calculated as if all members are encastré.

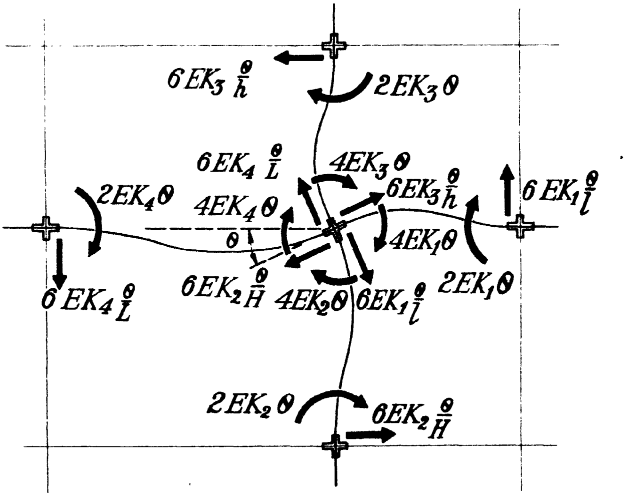


Fig. 182.

Next all joints are assumed to move small distances x_1, x_2, x_3, x_4, etc., horizontally and y_1, y_2, y_3, y_4, etc., vertically, and assumed to rotate through angles θ_1, θ_2, θ_3, θ_4, etc.

All these movements induce moments, thrusts and shears at the joints. At each joint the total vertical forces, the total horizontal forces and the total moment are all zero. Each joint has in general three movements and has three equations of equilibrium thus allowing us to calculate the values of x_1, y_1, θ_1, etc. The vertical movements are generally all assumed to be zero. In large frames the horizontal movements are also assumed to be zero and we have then only one movement and one equation for each joint.

To avoid confusion of signs we will take always the reaction of the member on the joint. The joints are shown in double lines in the diagrams.

Let Fig. 180 represent part of a frame and suppose the centre panel point moves a horizontal distance x, all other joints remaining fixed in position and direction. The movements and shears and the thrusts induced by this movement are shown in Fig. 181, where K_1, K_2, etc., are the stiffness factors and A_1, A_2, etc., the sectional areas of the members. The effect of a vertical movement of the central joint is readily seen by looking at Fig. 181 from the side of the page. Let Fig. 182 show the same frame where the central joint rotates through an angle θ. The induced reactions of the members on the joints are shown in the figure. If a member is hinged at its end the reactions induced by horizontal movement and by rotation are given in Figs. 183 and 184.

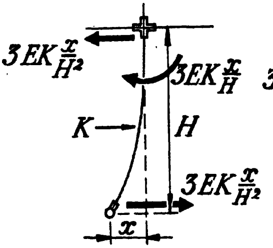


Fig. 183.

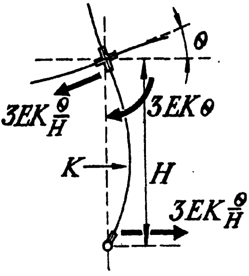


Fig. 184.

As an example of a frame with hinged ends we may take the two-hinged square arch in Fig. 185 carrying a point load out of centre. Under unsymmetrical loading the frame will sway sideways and it may be clearer to assume the frame moves first horizontally as in Fig. 186 and then the top joints rotate as in Fig. 187.

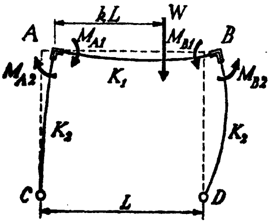


Fig. 185.

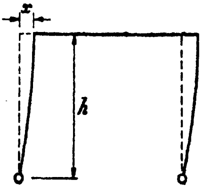


Fig. 186.

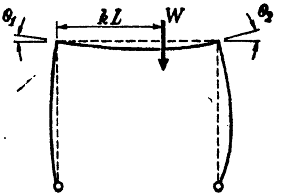


Fig. 187.

First assume that points A and B remain fixed in position and direction. The member $A\ B$ is then a simple encastré beam and :—

$$M_{A1} = W L k (1 - k)^2$$
$$M_{B1} = W L k^2 (1 - k)$$

Next assume the frame moves a distance x sideways the joints at A and B still remaining fixed in direction. This movement gives

$$M_{A2} = 3K_2 \frac{x}{h} \cdot E$$

$$M_{B2} = -3K_2 \frac{x}{h} \cdot E$$

If joint A rotates through an angle θ_1 this induces

$$M_{A1} = -4K_1 \theta_1 E$$
$$M_{A2} = -3K_2 \theta_1 E$$
$$M_{B1} = 2K_1 \theta_1 E$$

If joint B rotates through an angle θ_2 this induces

$$M_{A1} = 2K_1 \theta_2 E$$
$$M_{B1} = -4K_1 \theta_2 E$$
$$M_{B2} = -3K_2 \theta_2 E$$

These results have been set out at length but they may be tabulated at once as under

	M_{A1}	M_{A2}	M_{B1}	M_{B2}
No movement	$WLk(1-k)^2$	—	$WLk^2(1-k)$	—
Due to x	—	$3K_2 \frac{x}{h} E$	—	$-3K_2 \frac{x}{h} E$
Due to θ_1	$-4K_1 \theta_1 E$	$-3K_2\theta_1 E$	$2K_1 \theta_1 E$	—
Due to θ_2	$2K_1 \theta_2 E$	—	$-4K_1 \theta_2 E$	$-3K_2 \theta_2 E$

In this particular problem the horizontal thrust at the hinge C is equal and opposite to the horizontal thrust at the hinge D. Therefore the total value of M_{A2} must be the same as the total value of M_{B2}. The moments at joint A and at joint B must balance. Therefore :—

Total M_{A2} = Total M_{B2} . . . . (*a*)
Total M_{A1} + Total $M_{A2} = 0$ . . . . (*b*)
Total M_{B1} + Total $M_{B2} = 0$ . . . . (*c*)

Putting in the values from the table :—

$$3K_2 \frac{x}{h} E - 3K_2 \theta_1 E = -3K_2 \frac{x}{h} E - 3K_2 \theta_2 E \quad . \; . \; . \quad (a)$$

$$WL\,k\,(1-k)^2 - 4K_1 \theta_1 E + 2K_1 \theta_2 E + 3K_2 \frac{x}{h} E - 3K_2 \theta_1 E = 0 \quad (b)$$

$$WL\,k^2(1-k) + 2K_1 \theta_1 E - 4K_1 \theta_2 E - 3K_2 \frac{x}{h} E - 3K_2 \theta_2 E = 0 \quad (c)$$

Solving equations (*a*), (*b*) and (*c*) gives values for θ_1, θ_2 and x

$$\frac{x}{h} = \frac{(\theta_1 - \theta_2)}{2}$$

$$E\,\theta_2 = \frac{W\,L\,k\,(1-k)\,(-2K_1 + 1\tfrac{1}{2}K_2 - 2kK_1 - 3kK_2)}{(2K_1 + 3K_2)\,6K_1}$$

$$E\,\theta_1 = \frac{W L k\,(1-k)\,(4K_1 + 1\tfrac{1}{2}K_2 - 2kK_1 - 3kK_2)}{(2K_1 + 3K_2)\,6K_1}$$

Total $M_{A1} = W L k\,(1-k)^2 - 4K_1\,\theta_1\,E + 2K_1\,\theta_2\,E.$

Putting in the values for θ_1 and θ_2

$$M_{A1} = W L k\,(1-k)\,\frac{3K_2}{2(2K_1 + 3K_2)}$$

$$\text{and} \quad M_{B1} = M_{A1}$$

for a uniform load w covering $A\,B$

$$M_{A1} = M_{B1} = \frac{wL^2}{12}\,\frac{3K_2}{(2K_1 + 3K_2)}\,.$$

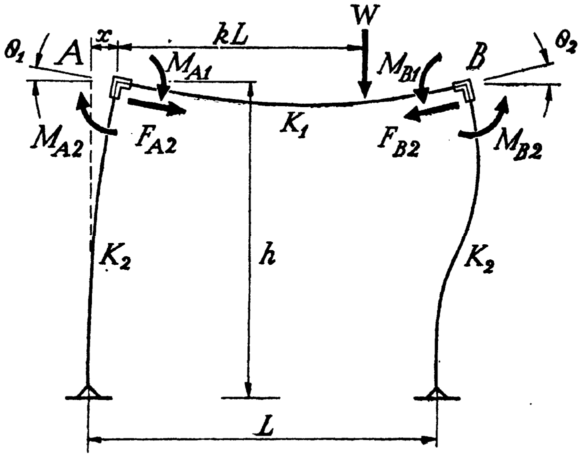


Fig. 188.

The encastré square arch in Fig. 188 is a little more complicated. We may tabulate the moments as under.

	M_{A1}	M_{A2}	M_{B1}	M_{B2}
No movement	$WLk(1-k)^2$	—	$WLk^2(1-k)$	—
Due to x	—	$6K_2\frac{x}{h}E$	—	$-6K_2\frac{x}{h}E$
Due to θ_1	$-4K_1\theta_1 E$	$-4K_2\theta_1 E$	$2K_1\theta_1 E$	—
Due to θ_2	$2K_1\theta_2 E$	—	$-4K_1\theta_2 E$	$-4K_2\theta_2 E$

We know that the total moment on joint A is zero and the total moment on joint B is zero, but we have three unknowns and must have another equation. We may use the fact that

$$F_{A2} = F_{B2}$$

	F_{A2}	F_{B2}
No movement	—	—
Due to x	$-12K_2\frac{x}{h^2}E$	$+12K_2\frac{x}{h^2}E$
Due to θ_1	$+6K_2\frac{\theta_1}{h}E$	—
Due to θ_2	—	$+6k_2\frac{\theta_2}{h}E$

Therefore $-12K_2\frac{x}{h^2}E+6K_2\frac{\theta_1}{h}E=+12K_2\frac{x}{h^2}E+6K_2\frac{\theta_2}{h}E$

$$\frac{x}{h}=\frac{\theta_1-\theta_2}{4}$$

The complete solution is

$$M_{A1}=\frac{WLk(1-k)K_2[K_1(13-2k)+K_2(4-4k)]}{(6K_1+K_2)(2K_1+4K_2)}$$

$$M_{B1}=\frac{WLk(1-k)K_2[K_2(11+2k)+4kK_2]}{(6K_1+K_2)(2K_1+4K_2)}$$

for a uniform load w covering AB,

$$M_{A1}=M_{B1}=\frac{wL^2}{12}\;\frac{2K_2}{2K_2+K_1}.$$

As an example of a large frame Fig. 189 shows part of a framed building with external spans all loaded and internal spans unloaded.

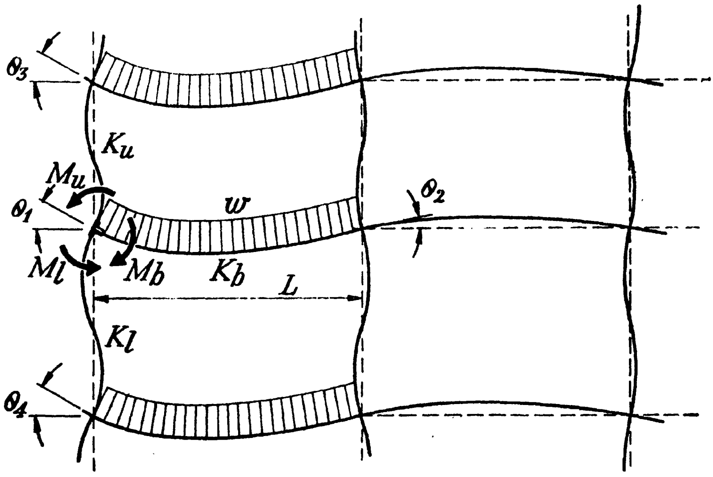


Fig. 189.

Suppose we wish to find the frame moments M_u, M_l and M_b at the outside joint. We may assume no sidesway in a large building and M_u, M_l and M_b are entirely dependent on θ_1, θ_2, θ_3 and θ_4.

	M_u	M_l	M_b
No movement	—	—	$wL^2/12$
Due to θ_1	$4\,EK_u\,\theta_1$	$4\,EK_l\,\theta_1$	$-\,4\,EK_b\,\theta_1$
Due to θ_2	—	—	$+\,2\,EK_b\,\theta_2$
Due to θ_3	$2\,EK_u\,\theta_3$	—	—
Due to θ_4	—	$2\,EK_l\,\theta_4$	—

To solve completely we must know the stiffness factor of every member in the whole building. For a practical drawing-office value we may assume (if the floor heights and spans are fairly regular) that $\theta_2 = \frac{1}{3}\,\theta_1$ and $\theta_3 = \theta_4 = \theta_1$.

Since $M_u + M_l = M_b$ we have at once

$$M_u = \frac{wL^2}{12}\,\frac{6K_u}{(6K_u + 6K_l + 3\tfrac{1}{3}K_b)}$$

$$M_l = \frac{wL^2}{12}\,\frac{6K_l}{(6K_u + 6K_l + 3\tfrac{1}{3}K_b)}\,.$$

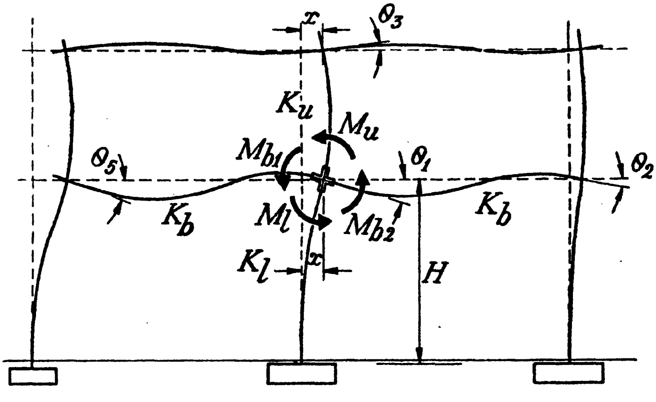


Fig. 190.

Again suppose Fig. 190 shows the lower part of a frame and that owing to shrinkage or temperature the joint has moved a distance x away from the centre line of the footing. Suppose we wish to find the value of M_l due to this movement.

	M_u	M_l	M_{b1}	M_{b2}
Due to x	—	$-6EK_l\frac{x}{H}$	—	—
Due to θ_1	$4EK_u\theta_2$	$+4EK_l\theta_1$	$4EK_b\theta_2$	$4EK_b\theta_1$
Due to θ_2	—	—	—	$2EK_b\theta_2$
Due to θ_3	$-2EK_u\theta_3$	—	—	—
Due to θ_5	—	—	$2EK_b\theta_5$	—

Also $\quad M_u + M_{b1} + M_{b2} + M_l = 0.$

For a drawing-office solution we may assume

$$\theta_3 = 0 \qquad \theta_2 = \tfrac{1}{2}\theta_1 \qquad \theta_5 = 3\theta_1$$

Putting in these values and solving we have

$$M_l = -6EK_l\frac{x}{H}\left[1 - \frac{4K_l}{(4K_u + 15K_b + 4K_l)}\right].$$

If $K_u = 250,\ K_l = 300,\ K_b = 200,\ H = 144$
$E = 2{,}000{,}000$ (all in inches and concrete units)
and if $\qquad x = 0{\cdot}04\%$ of 25′ 0″
then $\qquad M_l = 2{,}320{,}000$ lbs.-ins.

Actual Moments in Building Columns.—In an actual column, as usually built, there is a construction joint immediately below the main beam soffite and two more immediately above the floor. The column is thus weakest at the sections where the frame moments are a maximum. All columns, particularly outside columns, have heavy moments in theory due to frame action, unequal shortening of columns, shrinkage combined with foundation restraint, and relative temperature changes. Ninety-nine per cent. of all building columns so far built have been designed for a pure central load and no moment of any kind, yet the author has never yet seen any signs of distress in any column in a building due to frame action.

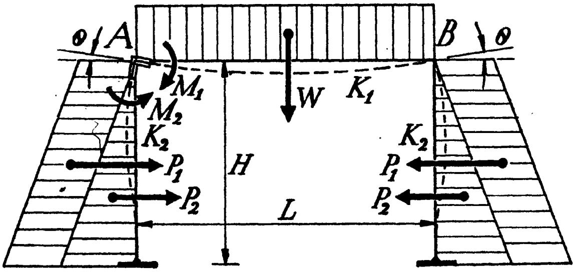


Fig. 191.

Box Culverts.—Let Fig. 191 represent in diagram a culvert carrying a load W uniformly distributed over the roof, a uniform earth pressure P_1 on each leg and a triangular earth pressure P_2 on each leg. For a triangular load on an encastré beam the reversed moment at the upper end is $\frac{P_2H}{15}$ and $\frac{P_2H}{10}$ at the bottom.

	M_1	M_2
No movement	$\frac{WL}{12}$	$\frac{P_1H}{12}+\frac{P_2H}{15}$
Due to θ at A	$-4K_1E\theta$	$+4K_2E\theta$
Due to θ at B	$+2K_1E\theta$	—

Obviously $M_1 = M_2$

$$\therefore \quad \frac{WL}{12} - 2K_1E\theta = \frac{P_1H}{12} + \frac{P_2H}{15} + 4K_2E\theta$$

$$E\,\theta\,(4K_2 + 2K_1) = \frac{WL}{12} - \frac{P_1H}{12} - \frac{P_2H}{15}$$

$$M_1 = \frac{WL}{12} - 2K_1E\theta$$

$$= \frac{WL}{12} - \frac{2K_1}{(4K_2 + 2K_1)}\left[\frac{WL}{12} - \frac{P_1H}{12} - \frac{P_2H}{15}\right].$$

Application of Square Arch.—The indiscriminate use of square arches in reinforced concrete is not to be encouraged as this form of construction is not really suited to the material. The occurrence of construction joints at the points of maximum stress and the uncertainty about the relative values of E and K, to say nothing of shrinkage, temperature and deflection of falsework during construction, make it necessary to allow a large margin over the calculated moments at all sections.

CHAPTER XX

CIRCULAR TANKS

Loading.—In all structures containing water at rest the applied loads can be determined exactly, the pressure varying directly as the depth from the free surface and being, at any point, the same in all directions. It follows therefore, that we may construct a tank which shall be circular in plan having walls devoid of all stiffness. For example, a bucket made of canvas remains stable when filled with water.

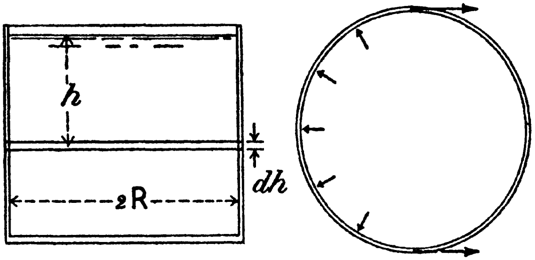


Fig. 192.

Let Fig. 192 represent a circular tank in plan and section.

Let $2\,R$ = diameter in feet
h = depth at any point in feet
w = weight of water per cubic foot = 62·5 lbs.

The pressure per square foot at any depth h is $w\,h$.

Take a small ring of tank of a height $d\,h$ and examine the forces on one-half of this ring. Since the tank walls are, by hypothesis, devoid of all stiffness there will be no reactions from similar rings immediately above and below and no bending moments in the ring itself. The internal stress on such a ring will obviously consist of a simple tension which, by symmetry, will be uniform all round the ring. Considering the equilibrium of the half-ring shown in plan, the only external forces are the water pressure and the two tensions.

Pressure per unit run of ring circumference $= w\,h\,(d\,h)$.
Tension in ring by a well-known theorem $= w\,h\,R\,(d\,h)$.

$$\text{Tension per unit height} = w\,h\,R.$$

In a tank 20 ft. 0 ins. diameter the tension in the wall per foot height would be, at a mean depth of 10 ft. 0 ins.,

$$= \frac{62{\cdot}5 \times 10 \times 20}{2} = 6250 \text{ lbs.}$$

Stresses.—Although, by our primary assumptions, concrete cannot be relied on to carry direct tension, it is clear from the behaviour of actual structures that the hoop tension is in fact taken by the concrete at all sections except where there is a vertical construction joint or a vertical shrinkage crack. We cannot guarantee that any tank will be free from vertical construction joints, as a breakdown in the concreting plant is always possible, nor can we, with modern cements, guarantee freedom from shrinkage cracks. It follows that we must supply hoop steel all round the tank capable of carrying the whole tension. In order to prevent fresh cracks forming in the sound concrete the hoop tension on the concrete must be kept below its tensile strength. This is rather a variable figure but working stresses of 200 to 300 lbs. per sq. in. may be allowed according to the richness of mix and standard of workmanship.

Where vertical joints or cracks have occurred, the whole of the hoop tension comes on the hoop steel. To prevent these joints or cracks opening and leaking, the tensile stress in the hoop steel should not exceed 12,000 to 13,500 lbs. per sq. in. in an unlined tank depending on the contents and the degree of tightness required. A stress of 16,000 lbs. per sq. in. may be employed in the walls of a tank having a plastic watertight lining. Whether or not the hoop steel assists the concrete in those parts where no cracks occur depends on the amount of shrinkage in the concrete. In a tank above ground with a waterproof lining, where the walls are absolutely dry, the concrete will shrink and there will be compression on the hoop steel and tension on the concrete when the tank is empty. When the tank is filled, the concrete stress may reach 200 to 300 lbs. per sq. in. before the hoop steel takes any stress. In an unlined tank below ground little or no shrinkage will occur and there will be no initial stress in the walls when the tank is empty. In this case we may assume that the stress in the hoop steel is always m times the stress in the concrete.

Suppose we have a 40 ft. diameter tank built above ground and lined with asphalte. At a depth of 30 ft. below water level the tension per ft. of height is

$$62{\cdot}5 \times 30 \times 20 = 37{,}500 \text{ lbs.}$$

The asphalte should be sufficiently elastic to cover any fine cracks and we may work to 16,000 lbs. per sq. in. in the hoop steel.

$$\text{Area of steel per ft. of height} = \frac{37{,}500}{16{,}000} = 2{\cdot}35 \text{ sq. ins.}$$

As the concrete will dry out and tend to shrink we should take all the tension on the concrete alone. The minimum thickness of wall at this depth, assuming a working stress of 300 lbs. per sq. in. on the concrete is then :—

$$300 \times 12'' \times T = 37{,}500,$$
$$\text{or} \quad T = 10{\cdot}4 \text{ ins.}$$

For a similar unlined tank underground the hoop steel at 30 ft. depth should be (per ft. of height)

$$\frac{37{,}500}{13{,}500} \quad \text{or} \quad 2{\cdot}77 \text{ sq. ins.}$$

The minimum thickness of wall at this depth (assuming $m = 15$) is given by

$$37{,}500 = 300 \times 12'' \times T + 300 \times 15 \times 2{\cdot}77$$
$$\text{or } T = 7{\cdot}0 \text{ ins.}$$

Watertightness.—With reasonably good materials and workmanship a 6 in. wall is non-porous. The only possible sources of trouble are bad joints, shrinkage cracks, and temperature cracks. There are on the market a great number of patent preparations for making "watertight" concrete. The author cannot quite see how any patent preparation mixed in with the concrete in small quantities can prevent dirty construction joints or shrinkage and temperature stresses. The only expedient worth considering is to increase the amount of cement (and possibly the sand) in the mix, and improve the supervision on the job.

Bottom Restraint.—Analysis.—Suppose we have a circular tank built of some uniform elastic material and suppose the walls may move about quite independently of the bottom, as shown in Fig. 193. If the walls are of uniform thickness T, the area resisting hoop tension per unit height of wall is T. The hoop tension at any depth h is $wh\,R$ and the hoop tensile stress is

$$\frac{wh\,R}{T}$$

The unit strain round the circumference and the unit extension of the radius at a depth h is $\frac{wh\,R}{T\,E}$, the total increase in length of the radius being $\frac{wh\,R^2}{T\,E}$.

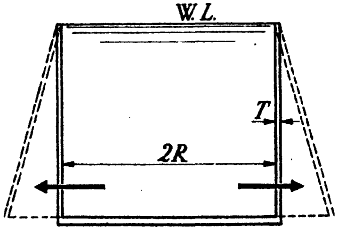


Fig. 193.

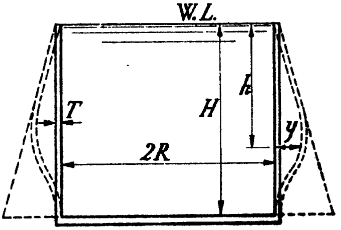


Fig. 194.

Now suppose that the lower end of the walls instead of being free to move is rigidly anchored both in position and direction to a rigid inextensible bottom. The walls will assume the shape shown in Fig. 194 the lower part forming, as it were, a series of vertical cantilevers attached rigidly to the bottom.

If y is the increase in radius at a depth h the unit strain round the circumference is $\frac{y}{R}$ and the unit hoop tension is $\frac{y\,E}{R}$. The total hoop tension per unit height is $\frac{y\,E\,T}{R}\cdot$ This hoop tension corresponds to a water pressure of $\frac{yET}{R^2}\cdot$ The actual water pressure is $w\,h$ and the unbalanced pressure $\left(wh - \frac{y\,E\,T}{R^2}\right)$ must be carried by the wall acting as a series of vertical cantilevers.

If we divide the wall into a series of vertical cantilevers by dividing the circumference into widths of one unit each, then we have

$$\frac{E\,I\,d^4y}{dh^4} = wh - \frac{y\,E\,T}{R^2}$$

where I is the moment of inertia of cross section of one unit cantilever

$$\frac{d^4y}{dh^4} = \frac{wh}{E\,I} - \frac{y\,T}{R^2I}$$

$$y = \frac{w\,h}{a^4EI} + \left[e^{\frac{ah}{\sqrt{2}}}\left(A\sin\frac{ah}{\sqrt{2}} + B\cos\frac{ah}{\sqrt{2}}\right) + e^{-\frac{ah}{\sqrt{2}}}\left(C\sin\frac{ah}{\sqrt{2}} + D\cos\frac{ah}{\sqrt{2}}\right)\right] \quad (1)$$

$$\text{where } a = \sqrt[4]{\frac{T}{R^2\,I}}$$

For an open-topped tank the values of A, B, C and D are found from the four conditions

$$\frac{d^2y}{dh^2} = 0 \text{ when } h = 0$$

$$\frac{d^3y}{dh^3} = 0 \text{ when } h = 0$$

$$y = 0 \text{ when } h = H$$

$$\frac{dy}{dh} = 0 \text{ when } h = H.$$

Still assuming that the walls are of uniform elastic material, the value of I per unit cantilever is $\frac{T^3}{12}$

$$a = \sqrt[4]{\frac{T \times 12}{R^2T^3}} = \frac{1{\cdot}87}{\sqrt{RT}}$$

By computing the values of y, $\frac{dy}{dh}$, $\frac{d^2y}{dh^2}$ and $\frac{d^3y}{dh^3}$ we have the deflection, slope, moment and shear at any point.

The important values are the shear and the bending moment at the base of the wall and these are given in Figs. 195 and 196.

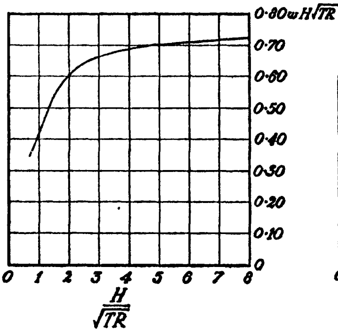


Fig. 195 shear.

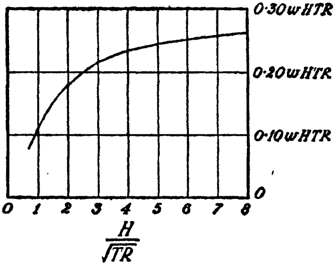


Fig. 196 moment.

Tapering Walls.—If the tank wall tapers uniformly from the water level to the base as in Fig. 197 we have

$$\frac{d^4y}{dh^4} = \frac{w}{EKh^2} - \frac{y}{h^2r^2}.$$

This cannot be solved for a general case although an approximate solution could be found for a given particular case. In practice large tank walls are built of a section similar to Fig. 200 and these may be treated as parallel walls the moments and shears being taken from Figs. 196 and 195.

Top Restraint.—If a tank has an inextensible and rigid roof the walls are restrained at the top. The moments and shears are obtained by modifying the conditions that control the constants in

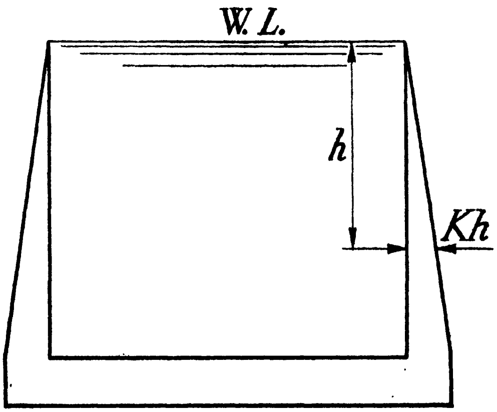


Fig. 197.

Approximate height of D in Fig. 198

$$= \frac{4820}{1880} \times 2 = 5{\cdot}12 \text{ say } 5 \text{ ft.}$$

Maximum hoop tension occurs at 25 ft. depth

$$= 62{\cdot}5 \times 15 \times 25 = 23{,}500 \text{ lbs.}$$

Tension on concrete (neglecting hoop steel)

$$= \frac{23{,}500}{12 \times 10} = 196 \text{ lbs. per sq. in.}$$

Hoop steel at 25 ft. depth and below

$$= \frac{23{,}500}{13{,}500} = 1{\cdot}74 \text{ sq. ins. per ft. height}$$

Cantilever moment from Fig. 196 $= 0{\cdot}265\, w\, H\, T\, R$

$$= 0{\cdot}265 \times 62{\cdot}5 \times 30 \times 0{\cdot}83 \times 15 \text{ lbs.-ft.}$$
$$= 74{,}500 \text{ lbs.-ins.}$$

$$A_s \text{ per ft. run of base of wall} = \frac{74{,}000}{16{,}000 \times 0{\cdot}86 \times 8{\cdot}5} = 0{\cdot}64 \text{ sq. ins.}$$

Fig. 199 shows the details.

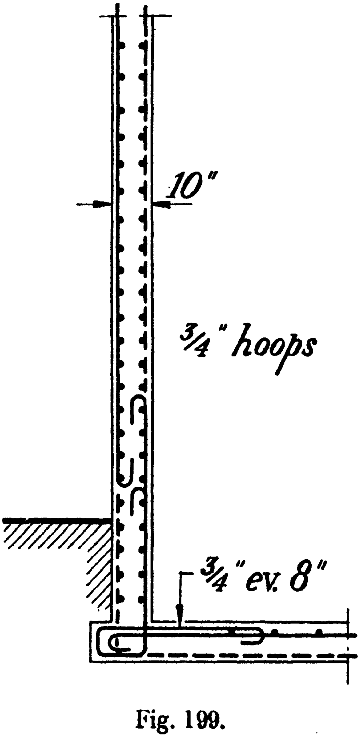


Fig. 199.

Application Shallow Tanks.—If the ratio $\frac{H}{\sqrt{TR}}$ is less than about 4·0 the tank may be regarded as "shallow." The theoretical moment and shear at the base of the wall are given in Figs. 196 and 195, but these do not give a complete picture of what happens in the upper part of the wall. It is possible, as shown in the next example, for the restraint to throw considerable hoop tension on the extreme top of the wall although the water pressure here is zero.

In small tanks the designer may readily arrive at covering values for the hoop steel, but for large tanks the deflections and moments should be calculated at length. Some designers advocate abolishing restraint in large shallow tanks by introducing a sliding joint at the base of the wall. Apart from the question of what happens under unequal external earth pressure, the author has not yet seen a detail of any such joint which seemed likely to function as intended.

Example.—Fig. 200 shows a gasholder tank 200 ft. 0 ins. diameter. The main part of the wall has a uniform thickness of 4 ft. 0 ins. and is uniformly reinforced with hoop steel having an area of 9 sq. ins. per ft. height of wall. Neglecting the outside earth pressure find the maximum tensile hoop stresses and the vertical steel required.

(The wall is thinned out at the extreme top but the effect of this on the main moments and shears is very small.)

$$\text{Ratio } \frac{H}{\sqrt{TR}} = \frac{40}{\sqrt{400}} = 2{\cdot}0 \text{ therefore the tank is not "deep."}$$

In any case, for such a large tank, it is worth while to spend a little time on the design. The moment and shear at the base of the wall from Figs. 196 and 195 are :—

$$\begin{aligned} M &= 0{\cdot}173\, w\, HTR = 0{\cdot}173 \times 62{\cdot}5 \times 40 \times 4 \times 100 \\ &= 175{,}000 \text{ lbs-ft. per ft. run of wall on plan.} \\ F &= 0{\cdot}6\, w\, H\, \sqrt{TR} = 0{\cdot}6 \times 62{\cdot}5 \times 40 \times \sqrt{4 \times 100} \\ &= 30{,}000 \text{ lbs. per ft. run on plan.} \end{aligned}$$

These values are not sufficient to design the whole wall and we must solve equation (1). Omitting the working we have :—

$$\left.\begin{aligned} A &= C = -\ 2{\cdot}95 \\ B &= +\ 1{\cdot}62 \\ D &= +\ 7{\cdot}52. \end{aligned}\right\} \times \frac{w}{a^4\,\mathrm{EI}}$$

By putting in values of 10 ft., 15 ft., 20 ft., 25 ft., etc., for h we arrive at the values of y, thus giving the shape of the wall. Taking the value of $\frac{d^2y}{dh^2}$ and again putting in values for h we have the moments. These are shown in Fig. 201. The tank is so wide compared with its depth that the restraint cantilever action entirely alters the stretching due to hoop tension. The amount of hoop tension at any point is proportional to the value of y. If we draw the line $A\ C$ in Fig.

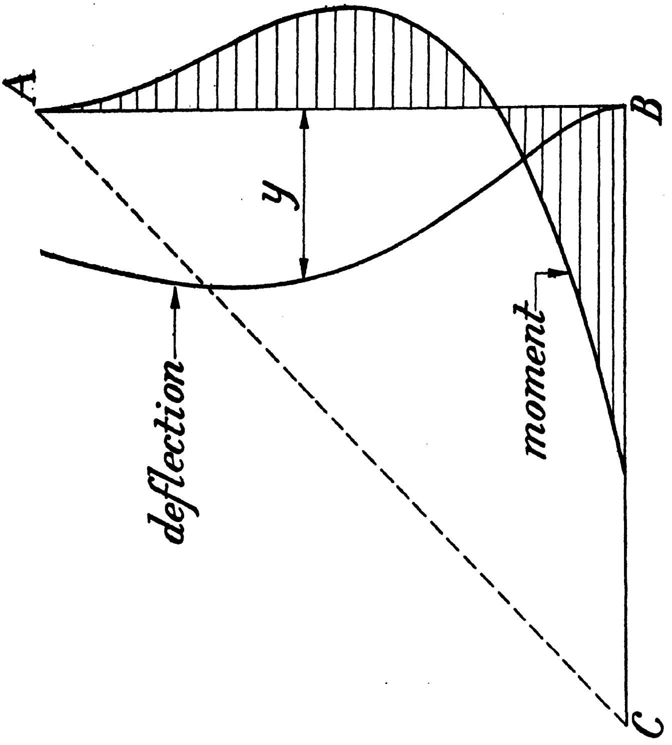


Fig. 201.

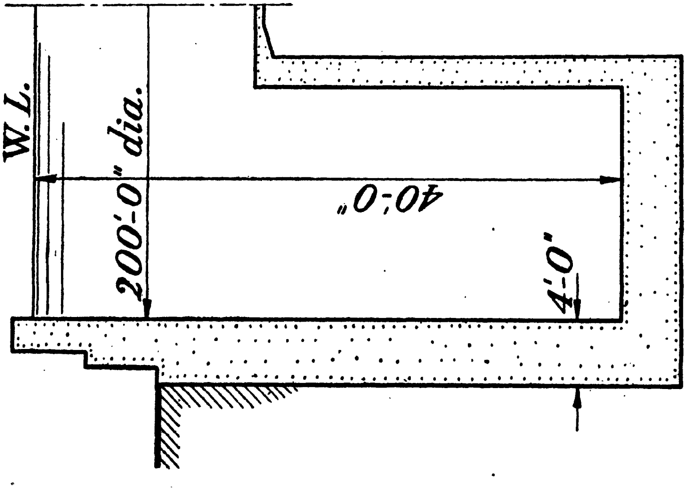


Fig. 200.

201 showing how the wall would stretch under hoop tension alone, if entirely unrestrained at the base, the distance $B\,C$, as already explained, is $\frac{w\,H\,R^2}{E\,T}$ and this distance represents to some scale the hoop tension due to a pressure $w\,H$. The maximum value of y is about $\frac{11{\cdot}5}{40} \times \frac{wHR^2}{E\,T}$ and the maximum hoop tension is therefore $\frac{11{\cdot}5}{40}\,wHR$ or $11{\cdot}5 \times 62{\cdot}5 \times 100$

$$= 72{,}000 \text{ lbs. on 1 ft. height of wall.}$$

Taking this on the steel alone, which is 9 sq. ins. area

$$\text{tensile stress} \quad = \frac{72{,}000}{9} = 8000 \text{ lbs. per sq. in.}$$

$$\text{On the concrete alone } \frac{72{,}000}{48 \times 12} \doteq 125 \text{ lbs. per sq. in.}$$

Assuming that the steel and concrete act together, the stress on the concrete

$$= \frac{72{,}000}{(48 \times 12) + (15 \times 9)} = 101 \text{ lbs. per sq. in.}$$

Main restraint moment = 175,000 × 12 = 2,100,000 lbs.-ins.
d for vertical cantilever say 45 ins.

$$\frac{M}{bd^2} = \frac{2{,}100{,}000}{12 \times 45^2} = 87 \text{ only,}$$

$$\text{vertical} \quad A_s = \frac{2{,}100{,}000}{16{,}000 \times 0{\cdot}88 \times 45} = 3{\cdot}3 \text{ sq. ins. per ft.}$$

say 1½-in. vertical bars every 6 ins.

Reversed moment half-way up the wall is about

570,000 lbs.-ins.

vertical A_s on outside of wall $= 0{\cdot}9$ sq. ins.

say 1⅛-in. vertical bars every 12 ins.

The theoretical hoop stresses look small, but practical questions must be considered. If the wall were not restrained at the base it would move out the distance $B\,C$ in Fig. 201 which is $\frac{w\,H\,R^2}{ET}$.

Taking E as 2,000,000 × 144 lbs. per sq. ft.

$$\frac{wHR^2}{ET} = \frac{62{\cdot}5 \times 40 \times 100 \times 100}{2{,}000{,}000 \times 144 \times 4} = 0{\cdot}0217 \text{ ft.}$$

$$= 0{\cdot}26 \text{ ins.}$$

The maximum value of y is only about 0·075 ins.

If the tank were standing in dry earth and were left empty to dry out, the shrinkage would approach 0·04% and the walls would tend to come in 0·04% of 100 ft. or 0·48 ins. ***which is more than six times the maximum value of y.***

On the other hand although our distribution of strength between the hoop steel and vertical cantilevers may be all wrong we have provided sufficient strength to take the whole of the water pressure and though the tank might crack it can scarcely fail. Assuming an equivalent fluid pressure of 15 lbs. per cub. ft. for the outside earth pressure the maximum hoop tension if all restraint fails is

$$(62{\cdot}5 - 15) \times 40 \times 100$$
$$= 190{,}000 \text{ lbs.}$$

On the concrete alone this is $\dfrac{190{,}000}{48 \times 12} = 330$ lbs. per sq. in.

On the steel alone $\dfrac{190{,}000}{9} = 21{,}200$ lbs. per sq. in.

The author considers the design quite safe if the tank is built on a really good foundation. If built on a doubtful or patchy site where unequal settlement may occur (however slightly) the thickness of wall and reinforcement should be increased.

Pre-stressed Hoops—Some modern circular tanks are built with pre-stressed steel. The tank is first built with a thin wall of plain or lightly reinforced concrete. The hoop steel is then fixed outside this wall, each bar being joined into a complete circle with turnbuckle. These are tightened until the bars have half their working stress, the concrete wall being correspondingly put under hoop compression. The hoop steel is then covered with an outer layer of concrete. When the tank is filled the steel reaches its full stress while the hoop compression on the concrete just disappears.

It is claimed that this eliminates tension cracks in the concrete even when high tensile stresses are worked to.

CHAPTER XXI

PILES AND SHEET PILES

Cast-in-Situ Piling.—Many systems of *cast-in-situ* concrete piles are on the market. Systems where the hole can be thoroughly examined before being filled may be safely recommended, but those where the hole is filled blindly or where the piles are likely to be disturbed during setting should be examined carefully before adoption. The reader is referred to the proprietors of the various systems for information.

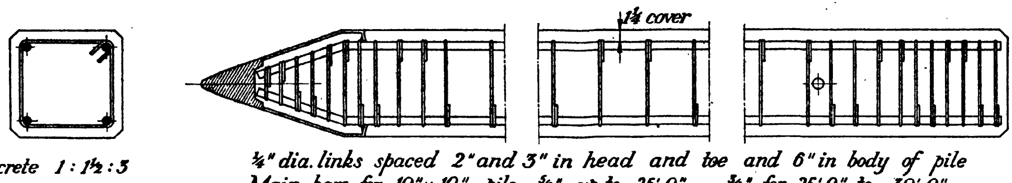
Pre-cast Piles.—The following are typical of everyday work :—

10″ × 10″ up to 25′ 0″ long, to carry 30 tons each.
12″ × 12″ up to 35′ 0″ long, to carry 45 tons each.
14″ × 14″ up to 40′ 0″ long, to carry 60 tons each.

Standard details from which some 100,000 piles have been made and driven are given in Fig. 202. Longer piles of these same sections may be used if the handling and driving are properly supervised. Longer piles are generally hollow. Fig. 203 shows a Williams hollow-pile.

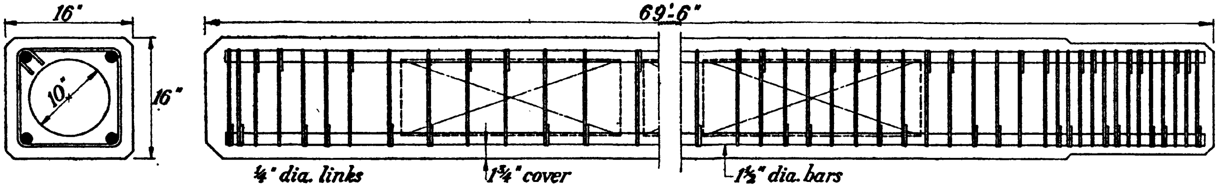
Handling Stresses.—On a small contract without continuous technical supervision the piles are rolled off the beds when a few days old. If the centre of the pile rolls over a bump a moment of $\frac{wL^2}{8}$ is produced. Rough handling is the chief cause of cracks in piles. When being pitched a strop is put round the pile about 3 ft. from the head and it is dragged over the site and hauled up to the frame. A moment of $\frac{w\,(L-3)^2}{8}$ may be expected during lifting. On a large contract where the piles are lifted under supervision by crane, if the piles are picked up by one strop placed $0{\cdot}296\,L$ from one end the moment is $0{\cdot}044\,wL^2$. If picked up by two strops each $0{\cdot}207\,L$ from each end the moment is $0{\cdot}022\,wL^2$. The reader must judge in each case what moment to allow for having regard to lifting appliances provided and the efficiency of the supervision.

To avoid *all* cracks the calculated tensile stress on the concrete, taking the section as plain unreinforced concrete, must not exceed 600 lbs. per sq. in. (This is a value found from practice.) To avoid serious and noticeable cracking the stresses calculated from Tables RB I to VII must not exceed 1000—1500 lbs. per sq. in. on the concrete and 25,000—30,000 lbs. per sq. in. on the steel. Piles



¼" dia. links spaced 2" and 3" in head and toe and 6" in body of pile
Main bars for 10" x 10" pile ⅝" up to 25'-0", ¾" for 25'-0" to 30'-0"
do. do. 12" x 12" pile ¾" up to 30'-0", ⅞" for 30'-0" to 35'-0"
do. do. 14" x 14" pile ⅞" up to 33'-0", 1" for 33'-0" to 40'-0"

Fig. 202.



Concrete 112 lbs. to 1·6 cu. ft. to 3·2 cu. ft.

Fig. 203.

longer than 50 ft. should, if possible, be made from normal-hardening cement, to avoid shrinkage cracks.

Pile Driving.—Many losses of energy occur during driving. The fall of a clutch-operated monkey is impeded by friction and the inertia of the rope and drum. The fall of a steam hammer is impeded by tight glands or a bent piston-rod. Energy is consumed by elastic deformation of the pile and by bounce of the monkey. These sources of loss are insignificant compared to the loss due to absorbtion of energy by the cushioning between the pile head and the monkey. For first-class driving the cushioning must be restricted to about 9 ins. of sound hard wood (elm or pyinkado) or 2 to 3 ins. of consolidated coco-nut matting. The monkey should weigh at least half as much as the pile and more if possible. When driving without a dolly the following version of the Dutch Rule may be used :—

$$\text{(Set in inches for 10 blows)} = \frac{20 \times (\text{weight of monkey in tons}) \times (\text{drop in feet})}{\left(1 + \dfrac{\text{weight of pile}}{\text{weight of monkey}}\right) \times \text{safe load in tons.}}$$

Drops of 3 ft. to 4 ft. 0 ins. are usual with monkeys of 2 to 6 tons according to the size of pile.

An elm dolly 10 ft. to 15 ft. long placed between the pile and the monkey absorbs about half the effective blow. When driving with a dolly the set given by the formula above should be halved.

The formula gives too high results for sets of less than ¼-in. for 10 blows and is probably incorrect for sets of more than 6 ins. for 10 blows but is a reliable indication of the resistance encountered during driving for sets of ½ in. to 2 ins. for 10 blows. In most grounds the resistance tightens if driving is interrupted for several hours or even several minutes, but in running sand it may decrease and piles should not be allowed to stay in running sand but should be driven or jetted through. Tightening up of the ground after a stoppage is not generally made use of to obtain a set but is regarded as part of the general factor of safety.

Strength of a Group of Piles.—The majority of piles pull up sharply near the end of their run and are held up by bearing and resistance near the lower end. Groups of four or six piles may be driven so that they are spaced a pile diameter apart, but groups of nine piles are best driven two diameters apart.

Continuous piling should be spaced out having regard to the general strength of the site. If the hard stratum in which the piles pull up or the strata immediately below are such that they will safely carry not more than 4 tons per square foot, piles carrying 64 tons each should not be spaced closer than 4 ft. centres in both directions. This provision only applies to large areas completely covered with uniformly spaced piles. When driving a large group of closely spaced piles, the inside piles should be driven first and the outside piles last.

Exploration of Site.—Before making piles on a new and unknown site test-piles should be driven and boreholes put down at least 10 ft. and preferably 20 ft. below the point of the lowest test-pile, to make sure that there is nothing soft below the pile-bearing stratum.

Pile Shoes.—When driving through clay devoid of stones no shoe is required. A flat-ended pile will drive just as well as a pointed one. For general use a light cast-iron point of about 30 lbs. weight is sufficient. When driving through very soft silt on to very hard rock a heavy blunt shoe of 100 to 150 lbs. is called for.

The Pile as a Column.—A deviation of 2 ins. out of position during driving may be expected and isolated piles must have tie-beams in both directions at the head. A pair of piles requires a tie in one direction, but groups of three or more may stand alone. As the piles carry heavy axial loads they must be relieved of all appreciable transverse load and if heavy horizontal thrusts have to be resisted, raking piles must be provided. Most pile frames can drive at a rake of 1 in 3 (some at a rake of 1 in 2) and 1 in 3 or 1 in 4 are usual slopes. The method of design when heavy transverse loads occur will be clear from the examples in Figs. 208 and 210 and Chapter XIII in the author's *Reinforced Concrete Arch Design*. Long piles in jetty work should be checked as long columns by the L.U.D. method (see *Civil Engineering*, March 1935).

Quantities in Piles.—The piles are cast horizontally and only three sides are shuttered. As the main bars have no laps the steel factor is 10·2 for these alone. Links ¼-in. diameter spaced at 6-in. centres in the body of the pile weigh 4 to 5 lbs. per yard of 12" × 12" or 14" × 14" pile. The shoes and lengths of gas-barrel for holes are separate items. The concrete is usually 1 : 1½ : 3 mix.

Example.—A pile is 14 ins. × 14 ins. × 40 ft. long. It has 4 bars 1 in. diam. and ¼-in. links at 6-in. centres in the body of the pile. It is used on a small job with no continuous technical supervision and is driven with a 2-ton monkey falling 3 ft. 6 ins. It has to carry a safe load of 60 tons. Give the quantities per yard of pile. Are cracks likely to appear and to what set should the pile be driven?

The quantities per yard of pile are :—

Concrete	$\frac{14 \times 14}{36 \times 36}$	=	$0{\cdot}151\ y^3$
Steel	$(3{\cdot}14 \times 10{\cdot}2 + 4\tfrac{1}{2})$ lbs.	=	·33 cwts.
Shuttering	$\frac{3 \times 14}{36}$	=	$1{\cdot}17\ y^2$

We may anticipate that the pile will be picked up 3 ft. from the head giving a moment of $200 \times \frac{37^2}{8} \times 12$ or 412,000 lbs.-ins.

Taking the pile as a plain concrete section, tensile stress on concrete

$$= 412{,}000 \times \frac{6}{14 \times 14^2}$$

$$= 900 \text{ lbs. per sq. in.}$$

Taking the section as Table RB IV

$$bd = 14 \times 12\tfrac{1}{4} = 172$$

$$bd^2 = 14 \times 12{\cdot}25^2 = 2{,}100 \text{ and } R = 197$$

$$A'_s = A_s = 1{\cdot}57 \text{ sq. ins.} = 0{\cdot}91\%.$$

From Table *RB* IV, $\frac{R}{f_c} = 0{\cdot}236$ and $f_s = 28 f_c$

$$\text{whence } f_c = 830 \text{ lbs. per sq. in.}$$

$$f_s = 23{,}100 \text{ lbs. per sq. in.}$$

The pile would certainly not develop serious cracks but might show fine hair cracks after lifting.

The set in inches for the last 10 blows is

$$\frac{20 \times 2 \times 3{\cdot}5}{\left(1 + \frac{3{\cdot}5}{2}\right) \times 60} \text{ or } 0{\cdot}85 \text{ inches, say } \tfrac{7}{8} \text{ ins.}$$

(This means 0·85 ins. for 10 blows or 0·085 ins. per blow.)

Sheet Piles.—Bearing piles carry heavy axial loads and little or no transverse load and are essentially struts. Sheet piles carry heavy transverse loads and little direct load and are essentially beams. While bearing piles are driven to a given set sheet piles are driven to a fixed depth.

Lateral Support.—The lower ends of the sheet piles are driven to such a depth that the counterpressure of the earth will safely hold the toes in position.

The upper ends may be tied back to any neighbouring structure. If no substantial structure is near, the upper ends may be tied back to anchor blocks or some type of dead-man or to an earth-plate. The earlier sheet-pile walls were usually of the type in Fig. 204. The piles are driven to a depth *k* below the lowest dredged level sufficient to hold the toes safely. The upper ends are enclosed by a beam which is tied at intervals to concrete anchor blocks. Judging by the large number of such lines of sheeting which have run forward, the anchorage is often insufficient or placed too close to the wall. The weight of the concrete block, plus the weight of the earth immediately above it, should be equal to the tension in the tie, and the block should be placed at a distance of at least 3*h* back from the wall on normal sites. The great difficulty is in judging the active and passive earth pressures. The earth behind the sheet pile is water-logged as no attempt is made to make the sheet piling water-tight. In addition, the upper part of the backing is usually filling while the lower part is virgin ground. The available counterpressure in front of the toe increases with the depth below the river bed and some idea of this variation may be gained by driving one or two trial piles on the line of the wall. Owing to the extreme difficulty of arriving at any definite computation we may take the simplest possible assumptions of a straight-line variation of pressure behind the wall and a straight-line variation of counterpressure in front, varying as the depth below the river bed. Referring to Fig. 204 if we assume that the active pressure is $\frac{1}{2}\,p_1\,(h+k)^2$ and the counterpressure $\frac{1}{2}\,p_2\,k^2$ and say that the moment of the counterpressure about the head of the pile must for safety be 1·5 times the moment of the active pressure then

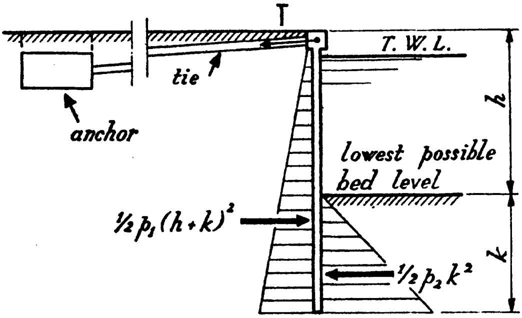


Fig. 204.

$$1{\cdot}5 \times \left(\tfrac{1}{2}\,p_1\,(h+k)^2 \times \tfrac{2}{3}\,(h+k)\right) = \tfrac{1}{2}\,p_2\,k^2\,(h + \tfrac{2}{3}k)$$

The values corresponding to this equation are given in Fig. 205.

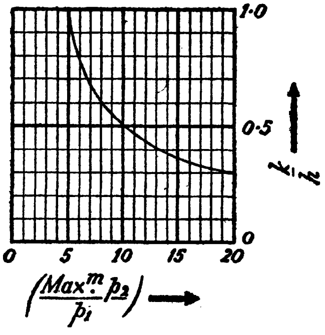


Fig. 205.

If we adopt Rankine's theory and assume that the ground is uniform in front of and behind the piling then

$$\frac{p_2}{p_1} = \frac{(1+\sin\phi)^2}{(1-\sin\phi)^2}.$$

As the ground is generally not uniform and as we do not know the correct value of ϕ, this expression is not very helpful. Even if we drive a few trial piles it is not easy for the beginner to interpret the data obtained. Nothing but experience and a well-developed engineering sense will enable the designer to come to a confident decision in any one specific case. Looking at the problem in its worst light we may regard the ground immediately behind the piles as water-logged to such an extent that it behaves as a fluid weighing about 100 lbs. per cub. ft. when the water in front of the piling is at top water level, giving a nett effective pressure for p_1 of about 40 lbs. per cub. ft. For a good bottom a penetration k in Fig. 204 equal to half h is found sufficient to hold the toe. From Fig. 205 this corresponds to a value of $\frac{p_2}{p_1}$ = about 10. If p_1 is 40 then p_2 must be about 400 lbs. per cub. ft. on a good site. On a soft site then k may equal h corresponding to $\frac{p_2}{p_1}$ = about 5 or if p_1 is 40 then p_2 must be about 200 lbs. per cub. ft. on a soft site.

Having chosen a value for p_1 and a value for p_2 we can calculate k in Fig. 204 (or take its value from Fig. 205). By taking moments we can easily calculate T and then find the maximum moment in the pile by finding the place where the shear force is zero.

Working Stresses in Sheet Piling.—Structural failures of sheet piling are very rare, but stability failures (or partial failures) are fairly common. A sound solution may be obtained by assuming a high value for the active pressure and a low value for the counter-pressure and taking high working stresses, always remembering that the structure is exposed to severe weather conditions. With a high estimate of the active pressure, stresses of 1000 to 1200 lbs. per sq. in. on the concrete (1 : 1½ : 3 mix) and 20,000 lbs. per sq. in. on the steel are allowable.

Example.—A line of sheet piling is 14 ft. 0 ins. high from top of bank to lowest level of river bed and is tied back 1 ft. 0 ins. below the top. The active pressure behind the wall is assumed as 30 lbs. per cub. ft. and the maximum counterpressure expected from the river bed is 200 lbs. per cub. ft. (i.e., 133 lbs. per cub. ft. allowing a factor of safety of 1·5).

Referring to Fig. 204 the value of $\frac{(\text{max}^{\text{m}}.\,p_2)}{p_2} = \frac{200}{30} = 6{\cdot}67$.

From Fig. 205 the depth to which the piles should be driven is about 0·7 × 14 ft. 0 ins. or say 10 ft. 0 ins. below river bed. The loading is shown in diagram in Fig. 206.

Checking the value of p_2 we have :—

$$\tfrac{1}{2}p_2 \times 10^2 \times 19{\cdot}67 = \tfrac{1}{2} \times 30 \times 24^2 \times 15$$

$$\text{actual } p_2 = 132 \text{ lbs. per cub. ft.}$$

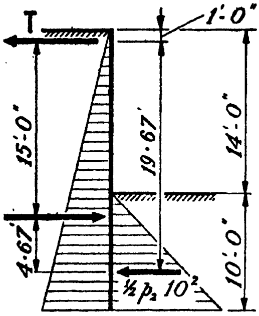


Fig. 206.

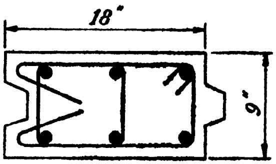


Fig. 207.

Taking moments about the point where the line of counter-pressure cuts the pile

$$T \times 19{\cdot}67 = \tfrac{1}{2} \times 30 \times 24^2 \times 4{\cdot}67$$

$$T = \frac{8650 \times 4{\cdot}67}{19{\cdot}67} = 2050 \text{ lbs. per ft. of wall.}$$

The shear force in the pile is zero at a depth of 11·7 ft. from the ground surface and the maximum moment is 2050 × 6·8 ft. or 167,000 lbs.-ins. This moment is per foot run of wall. If we use a sheet pile 18 ins. wide, the moment per pile is 252,000 lbs.-ins. Using a 9 ins. × 18 ins. sheeter with 1¼-in. cover outside the main bars the steel required each side will be about

$$\frac{252{,}000}{20{,}000 \times {\cdot}86 \times 7{\cdot}25''} \text{ or } 2{\cdot}02 \text{ sq. ins.}$$

Say 2 bars 1 in. = 1·57 sq. ins.
1 bar ⅞ in. = ·60 „ „

2·17 „ „

Checking f_c from Table RB IV, $A_s' = A_s = \dfrac{2{\cdot}17}{18 \times 7{\cdot}25}$ or 1·67%.

$$R = \frac{252{,}000}{18 \times 7{\cdot}25^2} = 265.$$

$$\frac{R}{f_c} = 0{\cdot}325, \text{ or } f_c = 820 \text{ lbs. per sq. in. only.}$$

A section of this pile is given in Fig. 207.

We might, of course, use an 8-in. pile with more steel or a 10-in. pile with less steel. The designer should make an estimate of the relative cost of the three different sizes before making a final choice.

Sheet Piles with Earth Plate.—If there is no substantial structure within reach of the sheet piling an earth-plate on piles offers a neater and surer anchorage than any form of dead-man. A typical example of the usual arrangement is shown in Fig. 208. The sheet piling is driven so that the heads are just above high water. On them stands a cast-in-situ wall connected to a horizontal earth-plate which rests also on two rows of inclined piles. The earth-plate transfers the horizontal pull from the upper end of the sheeting to the raking piles and the weight of the earth filling prevents the rear piles from pulling out of the ground. A pile will always offer considerable resistance to pulling out, but as the safe value of this resistance is unknown, it is usual to supply sufficient weight to keep the piles free from all tension, reckoning the resistance to withdrawal as a factor of safety. The calculation of the thrusts in the structure is

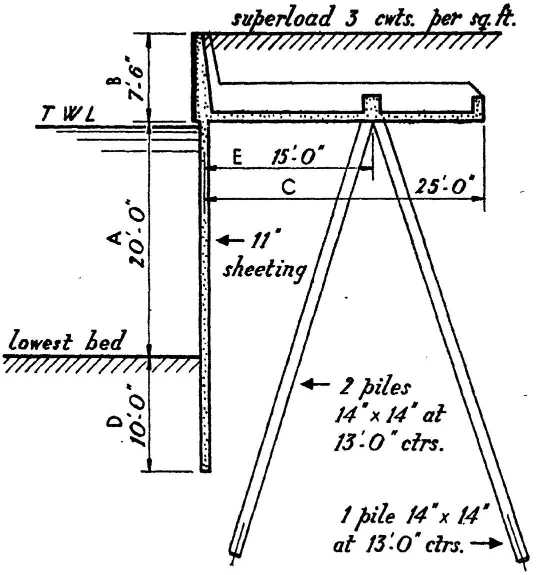

superload 3 cwts. per sq. ft.
B
7'-6"
T W L
E 15'-0"
C
25'-0"
A
20'-0"
← 11" sheeting
lowest bed
D
10'-0"
← 2 piles 14" x 14" at 13'-0" ctrs.
1 pile 14" x 14" at 13'-0" ctrs. →

Fig. 208.

not as simple as might be supposed. The estimated values of the earth pressures are complicated by the fact that the earth-plate shelters the sheet piles below it. How far down this sheltering effect is felt is a matter of guesswork, but we may draw an arbitrary line at 45° from the back edge of the plate and assume that the pressure above this line is proportional to the depth below the plate and the pressure below this line is proportional to the depth below the surface, including surcharge on the ground if any. A slope of 45° is possibly too steep for soft filling and a line at 30° may be drawn in extreme cases. The heads of the sheet piles are incorporated in the superstructure and the heads of the rakers are stripped and their reinforcement is left with a long lap into the earth-plate.

Some designers assume that the sheet piles are fixed in position and direction at the head. This is not so, as the whole earth-plate moves forward under horizontal load and this movement will reduce or reverse the fixing moment at the head of the sheet piles. The author has shown elsewhere that the thrusts in such structures may be found very closely by assuming that the piles are hinged at the heads and the main thrusts pass down the axes of the piles. When commencing to design such a structure we know the dimensions A and B in Fig. 208. If B is 0·4A or more, the dimension C may be taken as equal to A. If B is 0·3A or less then C may be taken as 1·25A. The dimension D on most sites is about 0·6A and E about 0·75A. These are not exact values, but will enable the designer to make a start.

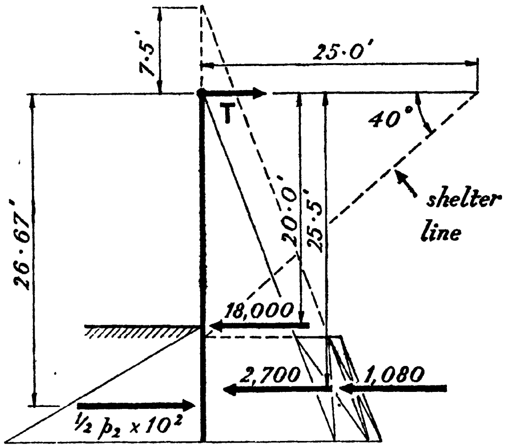


Fig. 209.

Example.—The structure in Fig. 208 is designed for an active earth pressure of 40 lbs. per cub. ft. The river bottom is estimated to give a counterpressure of 600 lbs. per cub. ft. (400 lbs. per cub. ft. with a factor of safety of 1·5). The shelter line may be taken at 40°. The wharf may carry 3 cwts. per sq. ft. surcharge. Check the design.

Taking the sheet piles first. We may assume them hinged at the top as far as the main thrusts are concerned. The pressures are shown in Fig. 209. On the back of the pile is a pressure of

$$\tfrac{1}{2} \times 40 \times 30^2 \text{ or } 18{,}000 \text{ lbs.}$$

due to the earth below the earth-plate. Below the shelter line there is additional pressure due to the fact that the full depth of earth becomes effective. This induces a further pressure of 7·5 × 40 over a height of 9·0 ft. or 2700 lbs. In addition we may get 3 cwts. per sq. ft. surcharge causing a further 3·0 × 40 × 9 or 1080 lbs.

Taking moments about the top point

$$\tfrac{1}{2}p_2 \times 10^2 \times 26{\cdot}67 = 18{,}000 \times 20 + 2700 \times 25{\cdot}5 + 1080 \times 25{\cdot}5$$

$$p_2 = 344 \text{ lbs. per cub. ft.}$$

Taking moments about the line of action of p_2

$$T \times 26{\cdot}67 = 18{,}000 \times 6{\cdot}67 + 2700 \times 1{\cdot}17 + 1080 \times 1{\cdot}17$$

$$T = 4660 \text{ lbs. per ft. run of wall.}$$

The method of finding the loads on the raking piles is given in Fig. 210. The thrust R_F on the front rakers is found by taking moments about the intersection point (*c*). The thrust on the back raker is found by taking moments about point (*a*). The direct axial loading on the sheet piles is found by taking moments about point (*b*). When finding the maximum and minimum thrusts we must remember that we must, for safety, be prepared to carry the maximum earth pressures and the maximum surcharge, but we cannot rely on these loads in cases where they tend to diminish a thrust. When calculating the maximum uplift on the back raker we must take the full tension T but cannot rely on the stabilizing action of the surcharge. When calculating the maximum downward load on the back raker we must take the surcharge into account, but we cannot rely on the relieving action of the full tension T and should count on, say, 50% of the maximum value. The earth pressures coming on the virtual back of the filling over the earth-plate are due to dry earth and have been run out on an assumption of 15 lbs. per cub. ft.

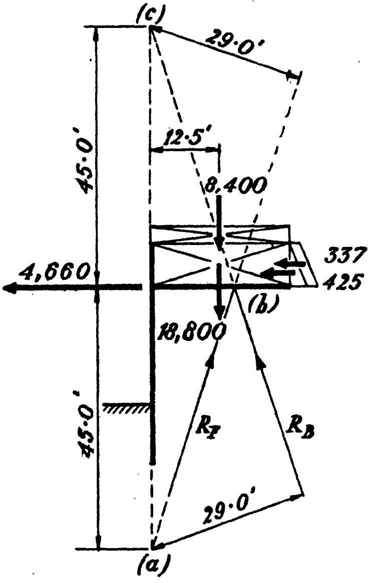


Fig. 210.

Taking moments about (*c*)

$$\text{Max}^{\text{m}}.\ R_F \times 29 = 4660 \times 45 + 18{,}800 \times 12{\cdot}5 + 8400 \times 12{\cdot}5 + 337 \times 41{\cdot}25 + 425 \times 42{\cdot}5$$

$\text{Max}^{\text{m}}.\ R_F = 20{,}040$ lbs. per ft. run of wall
= 118 tons on 13 ft. of wall or 59 tons per pile.

Taking moments about (*a*)

$$\text{Min}^{\text{m}}.\ R_B \times 29 = -4660 \times 45 - 425 \times 47{\cdot}5 + 18{,}800 \times 12{\cdot}5$$

$\text{Min}^{\text{m}}.R_B = +180$ lbs. per ft. run of wall

$$\text{Max}^{\text{m}}.\ R_B \times 29 = \text{say} -2330 \times 45 - 212 \times 47{\cdot}5 + 18{,}800 \times 12{\cdot}5 + 8400 \times 12{\cdot}5 - 168 \times 48{\cdot}75$$

$\text{Max}^{\text{m}}.\ R_B = +7475$ lbs. per ft. run of wall
= 44 tons per pile.

For safety a 14 in. × 14 in. pile is indicated.

The moment on the sheet piling itself is not clear as it depends on the fixing moment, if any, at the pile head. If we assume that the pile is hinged at the head the maximum moment occurs 15·3 ft. from the top where the shear is zero and is $4660 \times 15{\cdot}3 \times 0{\cdot}67 \times 12 = 575{,}000$ lbs.-ins. per ft. run of wall.

Alternatively if we assume that the pile is fixed in position and direction at the head and fixed by the earth at some arbitrary point, say 6 ft. below the river bed, the sheet piling is practically in the condition of a beam 26 ft. span fixed at both ends carrying a triangular load *W* where

$$W = \tfrac{1}{2} \times 40 \times 26^2 \text{ or } 13{,}500 \text{ lbs.}$$

The moments induced would be

$$-\frac{WL}{15} \text{ at the head}$$

$$-\frac{WL}{10} \text{ at the lower fixing point}$$

and $+\,{\cdot}0423\,WL$ between

$$-\frac{WL}{10} = 13{,}500 \times 26 \times \frac{12}{10} \text{ or } 420{,}000 \text{ lbs.-ins.}$$

A detail of the sheet piles is given in Fig. 211

$d = 9$ ins.
$bd = 18 \times 9 = 162$
$bd^2 = 18 \times 9^2 = 1460$

$A_s' = A_s = 5$ bars 1 in. diam. $= 3{\cdot}93$ sq. ins. $= 2{\cdot}43\%$.

From Table RB IV $\dfrac{R}{f_c} = 0{\cdot}416$

$f_s = 19{\cdot}5\, f_c$.

Taking a moment of 575,000 lbs.-ins. per ft. or 860,000 lbs.-ins. per 18-in. pile $R = 590$.

and $f_c = 1420$ lbs. per sq. in.
$f_s = 27{,}600$ lbs. per sq. in.

Taking a moment of 420,000 lbs.-ins. per ft. or 630,000 lbs.-ins. per 18-in. pile $R = 430$

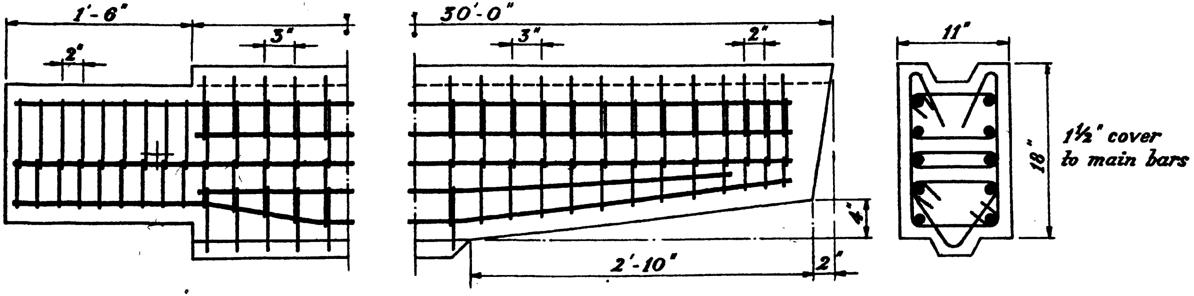
and $f_c = 1030$ lbs. per sq. in.

$f_s = 20{,}200$ lbs. per sq. in.

Whether these calculated stresses may be regarded as safe depends on the individual case and how nearly the earth pressure is expected to approach the maximum assumed value.

Interlock on Sheet Piles.—The details in Figs. 207 and 211 are excellent for piles of 10 ins. or 12 ins. thickness. For piles of 7 ins. or 8 ins. thickness, the fins are fragile and if the pile strikes a small obstruction and twists, they break off. This may be avoided by making the noses and recesses very shallow, say half an inch deep, but a small stone in the recess or a hard lump of clay may cause the piling to unlock as it is being driven. If the designer has the choice of a thin pile heavily reinforced or a thicker one lightly reinforced the latter is preferable as it gives a more satisfactory lock.

Handling Stresses, Quantities, Shoes.—In these respects sheet piles are similar to bearing piles.



NOTE all links ¼" diam. spaced at 2" and 3" centres in head and toe, and 6" centres in body of pile

Fig. 211.

CHAPTER XXII

HOPPER BOTTOMS

Shallow Bins.—A coal bunker may be looked on as shallow if its width is greater than one quarter its depth. Deeper bins may be treated as silos. Before we can design the hopper bottoms we must know the pressures coming on them.

Pressures in Shallow Bins—Sides.—The actual pressures in a coal bunker are vague and variable owing to the actions which take place consequent on the flow of coal. An accurate estimate of the pressures is, therefore, impossible. (In one case which came to the author's notice the designer inserted struts across a bunker near the bottom. His calculations showed clearly that these members were in compression. When the bunker was emptied the engineer had occasion to inspect the inside and was surprised to find that the "struts" showed unmistakable signs of having been subjected to heavy tension.)

Attempts have been made to apply Rankine's theory to bins having surcharged tops and to the problem of pressures on hopper bottoms. Such attempts are due to a misunderstanding of the premises on which the original theory was based and often lead to results which are both lengthy and incorrect.

As an example of fallacious application we may take the bins shown in Fig. 212. The bin contains coal having an angle of repose of 40° and weighing w per cubic foot. View I shows a section of the bin partly full, the top of the material sloping at 40°, and view II shows the bin filled up level with the top, the surface of the coal being horizontal. Now according to Rankine's theory for a surcharged slope the total pressure P_1 is equal to $\frac{w\,h^2}{2}\cos\Phi$ where Φ = the angle of repose = 40°, the pressure acting parallel to the surface. In case II where the surface is level the total pressure

$$P_2 \text{ is } \frac{w\,h^2}{2}\,\frac{(1-\sin\Phi)}{(1+\sin\Phi)}.$$

Putting in our values

$$P_1 = w \times \frac{15{\cdot}8^2}{2} \times {\cdot}766 = 96\,w$$

$$P_2 = w \times \frac{20^2}{2} \times \frac{(1-{\cdot}6428)}{(1+{\cdot}6428)} = 43{\cdot}5\,w.$$

B2

The horizontal component of $P_1 = 96\,w \times {\cdot}766 = 73{\cdot}5\,w$. By applying Rankine's theory, therefore, we see that the total pressure on the side in case I (even if we take only the horizontal component) is 73·5 *w* against 43·5 *w* in case II. The condition of loading in case II is brought about by adding two triangular volumes of coal *m n p* to the loading shown in case I. Therefore the addition of the two volumes of coal *m n p* causes an *inward* pressure of 30 *w* on the side. This is obviously absurd. The fallacy lies in the assumption that Rankine's theory for surcharged slopes applies to a bin. In order to comply with Rankine's conditions the surcharged slope must extend indefinitely. The conditions in a bin as case I approximate very closely to the conditions assumed by Rankine for a level surface. The heaped-up part may be looked upon as causing a superload above the level *n r* in the figure and the intensity of pressure at the foot of the wall may be taken as (*k w*) *h* where *h* is the average height measured to the mean surface *s t*, the total pressure being equal to the shaded area shown. In case III which shows a bunker having an abnormally short side such a procedure would give an excessive value for the lateral pressure. In case III, therefore, where the conditions approximate much more closely to Rankine's surcharged slope, the total horizontal pressure on the wall may be taken as $(k\,w)H^2$ where H is the height of the bunker wall. In intermediate cases the designer may calculate the pressure by both methods and take the lesser value.

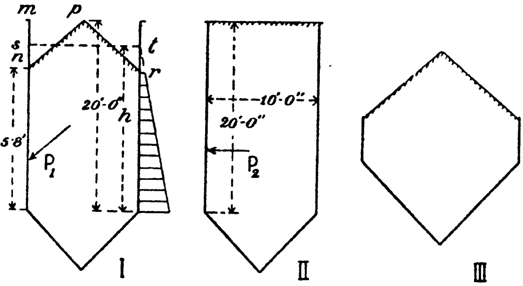


Fig. 212.

The value (*k w*) is the equivalent fluid pressure. A figure of 10 to 12 lbs. per cub. ft. for coal and 4 to 5 lbs. per cub. ft. for coke are in general use in commercial design.

Pressures on Sloping Bottoms.—There is one and only one definite fact regarding pressures in bunkers. The total vertical components of all pressures must equal the total weight of the contents. Any method of design which does not comply with this (there is at least one such in use) is wrong.

If we agree to take the pressures on the sides as horizontal, then the total vertical components of all pressures on the bottom must equal the total weight of the contents. We cannot be far wrong if we assume that the intensity of pressure on any sloping surface varies as the depth below the average free surface. In most cases then, if we know the *angle* at which the pressure is inclined to the sloping bottom we can find its intensity. It is practically certain that in all normal bunker work the direction of pressure on the sloping bottom lies somewhere between the vertical, as in Fig. 213, and a line at right-angles to the slope, as in Fig. 214, but probably nearer the latter as the coal tends to arch over the opening as indicated. If the pressure acts as in Fig. 213 the component normal to the slope is $w\,h\cos^2\theta$ and tangential to the slope is $w\,h\cos\theta\sin\theta$. If the pressure acts as in Fig. 214 the component normal to the slope is $w\,h$ and tangential to the slope nil. According to Rankine the normal and tangential pressures in a limitless mass under zero energy conditions are $(w\,h\cos^2\theta + (k\,w)\,h\sin^2\theta)$ and $(w\,h\cos\theta\sin\theta - (k\,w)\,h\cos\theta\sin\theta)$ respectively.

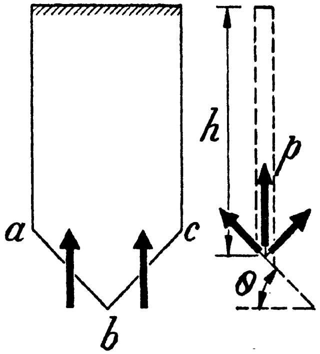


Fig. 213.

If h is 10 feet, $w = 56$ lbs. per cub. ft. and $(k\,w) = 12$ lbs. per cub. ft., the pressure normal to the slope if θ is 45° would be 280 lbs. per sq. ft., 560 lbs. per sq. ft., or 340 lbs. per sq. ft., according to which theory we adopt. In the author's opinion it is more likely to

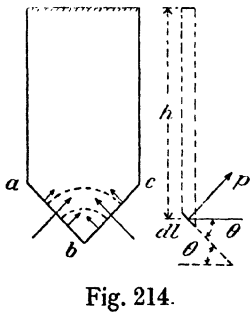


Fig. 214.

approximate to the largest of these three figures, particularly near the opening in a small bunker.

Types of Hopper Bottom.—Two types of bottom will be examined. The first type is continuously supported along two sides only, while the second type is supported all round.

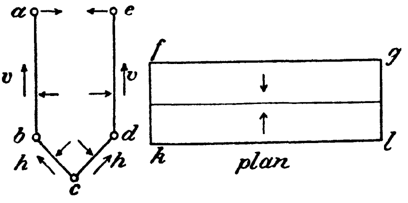


Fig. 215.

First Type.—Fig. 215 shows a plan and section of a bunker having a bottom sloping in two directions only, the section remaining the same throughout the whole length. (The arrows in plan show the direction of the slopes.) The section has been drawn as if the structure were pin-jointed at *b*, *c* and *d*. The admissibility of this assumption will be argued later. It has been assumed that the two top points *a* and *e* are tied together. Variations due to tying the vertical sides at other levels will be discussed later on. The four members *ab*, *bc*, *cd* and *de* together with the tie *ae* do not constitute a properly triangulated structure, even allowing for symmetry of loading. In order to keep the section in place we must introduce forces *v* and *h*. These forces are shear forces due to the fact that the side *ab* constitutes a vertical girder spanning from column to column and the bottom *bc* represents an inclined girder spanning from end to end of the bunker. (If the bunker is very long or if the bottom is very much cut up by holes, intermediate ties may be necessary.)

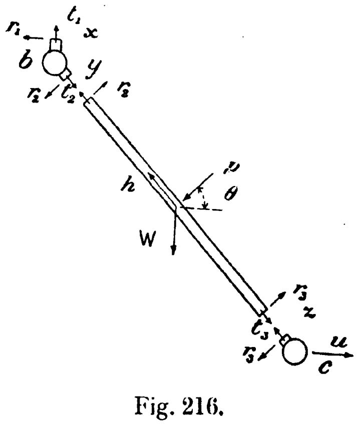


Fig. 216.

These two forces v and h may be assumed to act near the centres of the sides where the intensity of shear force in the girders would be greatest, i.e., half-way between b and c and half-way between a and b.

The side bc therefore has to span from b to c, has to resist a tension due to reactions from the sides ab and cd, and has to act as a girder.

This side is shown in detail in Fig. 216. Let W be the estimated self-weight of the member bc. The sections x, y and z are taken very close to the pin joints so that the bending moments in the members at the sections will be zero. The force u which represents the total reaction from the side cd on the side bc is, by symmetry, horizontal. The value of u is found by taking moments about b for the forces on bc. The shear forces r_1, r_2 and r_3 are easily found from the load on each member. In Fig. 215 the tie is drawn from a to e. If this tie is moved up or down the value of r_1 will alter. Taking the lowest part of the side zc and resolving in the direction bc we have the value of t_3.

Resolving horizontally for all the forces acting on xbc we have :—

horizontal component of $h = u - r_1 -$ (hor. comp. of p)

giving a value for h.

Also $$t_2 = t_3 - h + W \cos \theta.$$

Resolving vertically for the whole piece xbc

$t_1 =$ (vertical component of p) $-$ (vert. comp. of h) $+ W$.

For small hoppers the member bc would consist of a slab reinforced with bars at the bottom only, no bent-up bars being used. This being the case it appears that we ought to consider points b, c and d as pin joints without any fixing moments. Also since the steel runs right round the outside we may assume that the line of action of the applied tensions t_2 and t_3 coincides with the centre line of the tensile steel. The total area of steel required at any point, assuming we allow a stress of 16,000 lbs. per sq. in. for both direct tension stress and tension due to bending (there is no question of watertightness) is the sum of the areas required for tension and for bending calculated separately. For larger hoppers the slab would span horizontally and beams would be introduced on the underside. These would be made to coincide with the counterfort spacing. (If a cut design is required fixing moments may be allowed for at points b and c, some of the bars being bent up. The introduction of these moments can make little difference to the main reactions and tensions as calculated above. The sections at b and c may be examined as in Chap. VII, the direct tension being assumed to act at the lower layer of steel. The bent-up bars on the inside of the corners should be tied in with links.)

The centering for sloping bottoms is always awkward as the top boards require holding down. Any complication of the steel, particularly the introduction of top bars, should be avoided when possible.

Example.—A bunker is 15 ft. 0 ins. wide, 15 ft. 0 ins. deep and 40 ft. 0 ins. long. It contains coal weighing 56 lbs. per cub. ft. which exerts a lateral pressure of 12 lbs. per cub. ft. The hopper bottom slopes 45°. The sides are held by ties at the top, the surface being surcharged 5 ft. 0 ins. Design the hopper bottom under the following conditions :—

(A) Assuming that the pressure on the slope is at right-angles to the slope and working to stresses of 600 and 16,000 ($m = 15$).

(B) Assuming that the pressure on the slope is vertical and working to stresses of 750 and 18,000 ($m = 18$).

The left-hand side of Fig. 217 shows the loading for Case (A) and the right-hand side shows the loading for Case (B). The left-hand side of Fig. 218 shows the solution for Case (A) and the right-hand side shows the solution for Case (B). The values of the various reactions in Fig. 216 are as under.

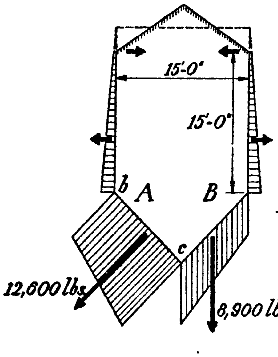


Fig. 217.

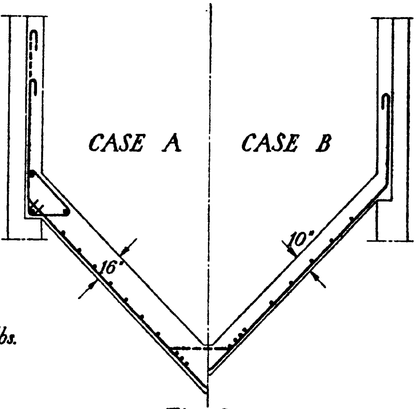


Fig. 218.

	Case (A)	Case (B)
p (normal to slope) . .	12,600 lbs.	6300 lbs.
p (tangential)	nil	6300 lbs.
r_1	1125 lbs.	1125 lbs.
r_2	6640 lbs.	3420 lbs.
r_3	7380 lbs.	3800 lbs.
u	10,400 lbs.	5350 lbs.
t_1	10,530 lbs.	5980 lbs.
t_2	8250 lbs.	5020 lbs.
t_3	7380 lbs.	3780 lbs.
h	520 lbs.	5980 lbs.
Min. d	14·0 ins.	8·3 ins.
A_s for moment in slab .	1·14 sq. ins.	0·88 sq. ins.
A_s for tension " " .	0·49 sq. ins.	0·24 sq. ins.
Total A_s " " .	1·63 sq. ins.	1·12 sq. ins.

This shows the difference between two methods of estimating the pressures. If we, in addition, assume arbitrary fixing moments at the corners in Case (B) we may further reduce the steel required for moment at mid-span. The values of *h* cause each slope to act as a deep girder and steel must be provided to resist the moment thus caused. Assuming that there are no cross-ties or cross-walls whatever in the length of 40 ft. 0 ins. the moment in Case (B) due to *h* is

$$5980 \times 40^2 \times \frac{12}{8} = 14{,}400{,}000 \text{ lbs.-ins.}$$

This is resisted by a girder about 11 ft. 6 ins. deep overall and 10 ins. wide.

$$bd^2 = \text{say } 10 \times 130^2 = 169{,}000$$

$$R = \frac{14{,}400{,}000}{169{,}000} = 85$$

$$\begin{aligned} jd &= \text{say } 118 \text{ ins.} \\ bjd &= 1180 \text{ sq. ins.} \\ \text{Total shear} &= 119{,}600 \text{ lbs.} \\ \text{Shear stress} &= 100 \text{ lbs. per sq. in.} \end{aligned}$$

(This is safe on such a deep girder.)

A_s for moment in girder

$$= \frac{14{,}400{,}000}{18{,}000 \times 118} = 6{\cdot}8 \text{ sq. ins.}$$

If we take a design half-way between the two extreme assumptions in Case (A) and Case (B) a sound yet reasonably cheap structure will result.

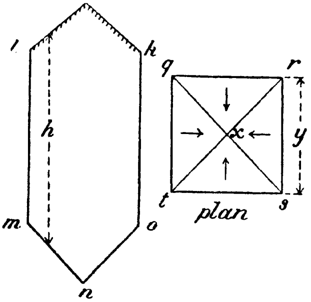


Fig. 219.

Second Type.—This is used for bunkers which are square or approximately square in plan. The hopper bottom slopes four ways as shown by the arrows in plan in Fig. 219. The pressure on the bottom is found as for the first type. Most bottoms are reinforced with bottom bars only, in both directions and so we may look upon each triangle of bottom *q x t*, *t x s*, *s x r* and *r x q* as being simply supported along all its edges. One edge is suspended from the vertical sides while the other two edges are held by horizontal tensions somewhat similar to the reaction u in Fig. 216 supplied by the adjacent triangles. The minimum slope of such a hopper bottom should be not less than 45° or the angle formed by the two planes where they meet at the diagonal will be too small to be effective. All the tension is eventually carried to the foot of the vertical sides. Since the pressures on the vertical walls are horizontal, the sum of all the vertical tension must equal the weight of coal in the bunker plus the self-weight of the bottom. The intensity of tension along the foot of the vertical wall will probably vary, being greater at the middle of the sides than near the corners.

For practical purposes it may be taken as uniform. The stresses in each triangle of the bottom, considering that each may act as a girder in its own plane and also as a slab spanning both ways, are more or less a matter of conjecture. As far as the bending stresses for the slab are concerned we can show (by methods similar to those in Chapter XVII) that the moments in an equilateral triangle, simply supported on all sides and carrying a uniform pressure p are $\frac{pl^2}{72}$ in all directions where l is equal to the side.

With a square bunker having a hopper sloping at 45°, the true shape of the four triangles is not far from equilateral and a moment of

$$\frac{p_n \times (\text{bunker side})^2}{72}$$

may be designed for, where p_n is the pressure normal to the slope at the centre of gravity of the triangle.

We may take the direct tension as acting along the centre-line of the steel. Therefore the area of steel at any point is the sum of the areas required for bending and direct tension calculated separately. As the position and direction of the maximum bending stress are somewhat vague it is best to be on the safe side and assume that the greatest tension, which we may take as the vertical tension already calculated at the foot of the sides (compare the tensions in the example of the first type) and the greatest moment occur together.

Looking at the problem from another point of view. If the hopper bottom were devoid of all stiffness but were capable of resisting tension and were left to take up its own shape as a suspended member it would not depart far from the shape shown in Fig. 219. It would appear therefore, from general principles, that the moments produced cannot be very large.

The method of design is therefore to estimate the total weight of coal and the self-weight of the hopper bottom. Divide this by the perimeter at the base of the vertical walls. This gives the direct tension. Design the slab of the bottom to resist a pressure corresponding to the average height of coal h as shown in Fig. 219 taking a bending moment of $p_n \times \frac{(\text{side})^2}{72}$ and add an area of steel sufficient to take the direct tension.

The sides of such bunkers usually span horizontally and are independent of the bottom as far as their bending moments are concerned.

Example.—A bunker is 15 ft. 0 ins. square and has a vertical wall 20 ft. 0 ins. high. The hopper bottom slopes at 45°. The coal weighs 56 lbs. per cub. ft. with a lateral pressure of 12 lbs. per cub. ft. The top surface is surcharged at 35°. Design the hopper bottom for stresses of 750 and 18,000 for the two following conditions.

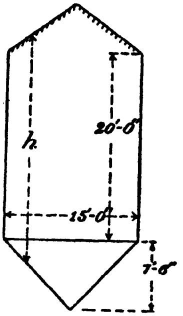


Fig. 220.

Case (*a*) Assuming the pressure is normal to the slope

Case (*b*) Assuming the pressure is inclined at the angle given by Rankine.

An outline of the bunker is given in Fig. 220.

$$\text{Volume of coal} = 15 \times 15 \times (20 + \tfrac{1}{3}(7'\ 6'' + 5'\ 2''))$$
$$= 5450 \text{ cub. ft.}$$

$$\text{weight} = 5450 \times 56 = 305{,}000 \text{ lbs.}$$

$$\text{Self-weight of bottom say} = 22{,}000 \text{ ,,}$$

$$327{,}000 \text{ ,,}$$

$$\text{tension per ft. run of wall} = \frac{327{,}000}{4 \times 15} = 5470 \text{ lbs.}$$

Average height $h = 24{\cdot}25$ ft.

Pressure normal to slope Case (*a*) $= 24{\cdot}25 \times 56 = 1360$ lbs. per sq. ft.

Pressure normal to slope Case (*b*)

$$= 24{\cdot}25 \times 56 \times \tfrac{1}{2} + 12 \times 24{\cdot}25 \times \tfrac{1}{2}$$
$$= 830 \text{ lbs. per sq. ft.}$$

Component of self-weight of bottom say 70 lbs. per sq. ft.

$$\text{Moment Case } (a) = \frac{(1360 + 70) \times 15^2}{72} \times 12$$

$= 53{,}500$ lbs.-ins.

Moment Case (*b*) $= \dfrac{(830 + 70) \times 15^2 \times 12}{72}$

$= 33{,}800$ lbs.-ins.

Min^m. d Case (*a*) $=$ 5·68 ins. say 7½ ins. overall

Min^m. d Case (*b*) $=$ 4·51 ins. say 6 ins. overall

A_s due to direct tension $= \dfrac{5470}{18{,}000} = 0{\cdot}302$ sq. ins.

A_s due to bending Case (*a*) $=$ 0·572 sq. ins.

A_s due to bending Case (*b*) $=$ 0·486 sq. ins.

Total A_s Case (*a*) $=$ 0·874 sq. ins. (¾-in. every 6 ins.)

Total A_s Case (*b*) $=$ 0·788 sq. ins. (⅝-in. every 4½ ins.).

By taking an average between Case (*a*) and Case (*b*) a sound yet economical design may be found.

The spacing given is the spacing of the bars as they run down the slopes. The bars will run both ways right across the bottom and each bar will be a different shape and length, having two vertical pieces at the ends, a horizontal piece in the middle and two sloping pieces. They require careful measuring and bending as they are " tight " in the centering. This may be avoided by making each bar in two halves and lapping near the centre but this does not make such a sound job. The spacing of bars where they run across the slopes will be about $\sqrt{2}$ times the spacing down the slope but this is correct as there is much less direct tension in this direction.

In theory it is possible, by putting in top bars at the intersections, to reduce the positive moments by introducing reversed moments. In practice top steel in hoppers should always be avoided.

Mixed Types.—Occasionally a mixture of the types crops up. Weak points may arise where a slab, acting as a girder, tapers to a point. In complicated cases the minor slopes may be omitted from the reinforced concrete structure and be put in afterwards in mass concrete.

Some American authorities advocate curved bottoms, which are in tension only, but the centering costs are prohibitive (see Chapter XXV for the method of treatment).

CHAPTER XXIII

HOLLOW TILE FLOORS

The floors in modern steel-frame buildings are invariably fire-proof and generally consist of some kind of tile and concrete construction. The tiles may be either terra-cotta, asbestos, pressed-steel or concrete and very many patent types are on the market. The underlying idea is common to them all. For floor slabs of large span a solid concrete floor is very heavy and forms a large percentage of the total load to be carried. In consequence a large proportion of the bending strength of the slab is used up in carrying its own self-weight and is not available for carrying useful load. Moreover, the concrete below the neutral axis in mid-span, being in tension, serves no useful purpose. If, therefore, we can replace this concrete by some light and cheap material we shall save both weight and cost. If in addition the material does away with all or part of the centering a further saving is effected. The problem might be solved by using a series of small T-beams but the complications of the centering counteract the saving in material and the projecting ribs make the ceiling treatment more difficult. It is not intended to give a résumé of all the different types of floor in use and one standard type only will be described. The tiles in this case are terra-cotta and 12 ins. × 12 ins. in plan. The height of tile varies by even inches from 4 ins. to 12 ins. The thickness and weight vary slightly with the clay employed in their manufacture. For floors spanning one way these tiles are open at the ends.

If the makers are not able to supply details of the weights of their tiles the designer may assume that they weigh 120 lbs. per cub. ft. and are ·75 ins. thick for the smaller sizes and ·9 ins. thick for the larger sizes.

For open-ended tiles the following are average weights:—

12″ × 12″ × 4″ tile 17 lbs. each.
12″ × 12″ × 6″ „ 25 lbs. „
12″ × 12″ × 8″ „ 33 lbs. „
12″ × 12″ × 10″ „ 39 lbs. „
12″ × 12″ × 12″ „ 44 lbs. „

It will be seen that open-ended tiles weigh about 45 or 50 lbs. per cub. ft. of total space occupied against 150 lbs. per cub. ft. for concrete. The tiles should be uniform and regular in size, and strong enough to bear a man's weight. At the present moment prices are fluctuating and a comparison of cost is not possible. The designer must take current prices. With concrete at 40/- per cub. yd. and 12 ins. × 12 ins × 6 ins. tiles at £25 per thousand laid on centering, a cubic foot of concrete costs 1/6 and a cubic foot of tile costs 1/0. With these figures the substitution of one cubic foot of tile for one cubic foot of concrete saves 100 lbs. in weight and about 6d. in cost.

For fairly regular rectangular bays of floor these tiles require one board under each rib. For odd-shaped bays it will be found more convenient to close-centre the whole bay. Three typical sections are shown in Fig. 221. Arrangement I is a light roof section either for a small span flat or a mansard slope. Arrangement II is a medium floor section and arrangement III a long span floor section. The strength of the tiles themselves will vary with the make. In arrangements II and III if the neutral axis falls below the top of the tile the distance c may be increased by ½-in. to allow for the resistance of the tile in compression. In these two cases the compressive stress on the tile will be well below 750 lbs. per sq. in. In arrangement I where the tile runs up to the compression edge it is safer to design the ribs as rectangular beams and neglect the tiles, in which case a top bar may be added. Arrangements II and III are calculated as T-beams with a flange thickness equal to c + ½-in. The concrete cover over the tiles c should not be less than 1 in. and preferably not less than 1½ ins. The thickness of the concrete ribs r should not be less than 2½ ins. for arrangement II and not less than 3½ ins. for arrangement III and preferably not less than 3 ins. and 4 ins. respectively. In II which has only one bar per rib the bars may be bent as shown in the longitudinal section the bars in one rib being taken over the left-hand supports and those in the next rib over the right-hand supports and so on. In arrangement III the bars may each have one end bent up, or one bar may be straight and the other bent up at both ends.

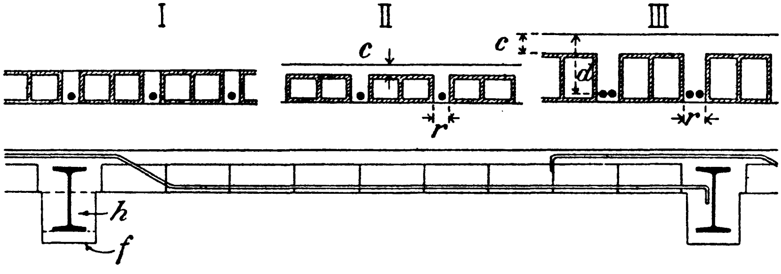


Fig. 221.

Design.—Hollow-tile floors are used extensively for light superloads (112 lbs. per sq. foot downwards). They are calculated for moments of $\frac{wl^2}{12}$ and $\frac{wl^2}{10}$ for central and end spans respectively. As a rule only the mid-span section is examined for bending, the support section being checked for shear. The resistance of the flange in compression is given in Table HT I, t being equal to $c + \frac{1}{2}$-in. In choosing between arrangements II and III the size of bar should be considered, ¾-in. or ⅞-in. being taken as the largest permissible size. Bars of 1 in. or over are rather difficult to deal with in shallow floors. The cover to the bars need not exceed ½-in. as the floors are unlikely to be exposed to damp and the ceiling is, as a rule, plastered on below the tiles. Should the reader like to know the value of $j\ d$ more closely, the values in Fig. 222 which shows the relation between the lever arm $j\ d$ and the ratio $\frac{t}{d}$ may be used. The figure has been drawn on the assumption that the neutral axis occurs at 0·429 the depth. An approximate value of the effective depth may be found by using $R = 115$ to 138 for stresses of 750 and 18,000. As far as bending moment and shear force are concerned the self-weight per square foot is the self weight at mid-span as the extra concrete near the beams does not affect the stresses in the floor slab. In estimating the weight of the floor slab for a preliminary design we may take the weight as 85 to 100 lbs. per cubic foot on the overall depth. Adjustments must be made later when the size of tile, thickness of rib and diameter of bars is finally settled. The reader should work out several values for floors and keep them for reference. It is generally not possible to work to standard sections as each job has its own peculiarities, but worked out examples are always useful for comparison. Tiles 4 ins., 5 ins. and 6 ins. high are the most frequently used with $c = 1\frac{1}{2}$ ins. or 2 ins.

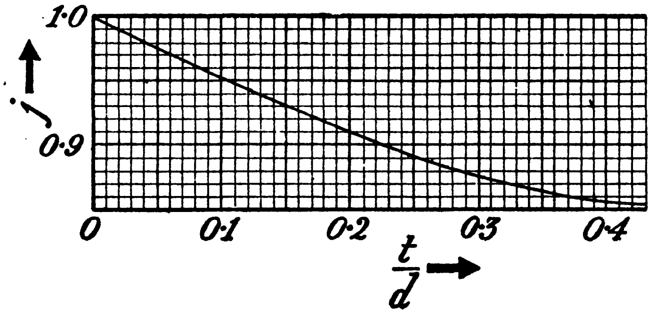


Fig. 222.

Example.—A hollow tile floor spans 12 ft. 0 ins. and carries a superload of 100 lbs. per sq. ft. The finishes are estimated at 16 lbs. per sq. ft. in addition. Design one interior panel, stresses not to exceed 750 and 18,000.

Load = super	100 lbs. per sq. ft.
finishes	16 „ „ „ „
Self-weight say 6″ at 95 lbs. =	48 „ „ „ „
	164 „ „ „ „

$$M = 164 \times 12^2 \times \frac{12}{12} = 23{,}600 \text{ lbs.-ins. per foot run}$$

$$\text{at } R = 130, \ d = \sqrt{\frac{23{,}600}{130 \times 12}} = 3{\cdot}89 \text{ ins.}$$

Considering the length of span, a floor with 4-in. tiles and 1½-in. concrete cover, giving a 5½-in. total thickness seems indicated.

Making the ribs 3 ins. wide the actual weight per sq. ft. at mid-span

1½ ins. concrete cover =	19 lbs.
One rib 3 ins. × 4 ins. at 15 ins. centres =	10
One tile 12 ins. × 12 ins. × 4 ins. at 17 lbs. at 15 ins. centres =	14
	43
Superload and finishes	116
	159 lbs.

$$\text{Actual bending moment} = 159 \times 12^2 \times \frac{12}{12}$$

$$= 22{,}900 \text{ lbs.-ins. per foot.}$$

(Notice the very small difference made by substituting the actual weight for the estimated weight.)

Using ⅝-in. cover and assuming ¾-in. bars.

$d = 5\frac{1}{2}$ ins. − 1 ins. = 4·5 ins.

The thickness of flange in compression is 1½ ins. concrete + ½-in. allowance for tile ∴ the neutral axis falls inside the slab and the floor has the same bending strength as a solid slab.

$$A_s = \frac{22{,}900}{18{,}000 \times 0{\cdot}857 \times 4{\cdot}5} = 0{\cdot}33 \text{ sq. ins. per foot.}$$

Area per rib (with ribs at 15 ins. centres)

$$= 0{\cdot}33 \times \frac{15}{12} = 0{\cdot}413 \text{ sq. ins.}$$

Say one bar ¾-in. per rib.

[A section across the floor will be similar to detail II in Fig. 221, with $r = 3$ ins. and $c = 1\frac{1}{2}$ ins.]

Maximum shear at support = 159 × 6 lbs. per ft.

$$= 159 \times 6 \times \frac{15}{12} \text{ lbs. per rib} = 1190 \text{ lbs.}$$

Shear area $b\,j\,d$ of rib = 3 × 0·857 × 4·5 = 11·5 sq. ins.

$$\text{Intensity of shear} = \frac{1190}{11{\cdot}5} = 104 \text{ lbs. per sq. in.}$$

If the rib does not start until 6 ins. away from the support the shear would be about 95 lbs. per sq. in.

With bent-up bars a value of 100 lbs. per sq. in. is safe.

A sounder design would be to make the ribs 4 ins. wide and put two bars $\frac{9}{16}$-in. in each.

A floor spans 18 ft. 6 ins. wall to wall as an isolated span. It carries a superload of 84 lbs. per sq. ft. and finishes estimated at 14 lbs. per sq. ft. Design this floor. Stresses not to exceed 750 and 18,000.

Load = superload	84 lbs.
finishes	14 „
Estimated self-weight say 10 ins. at 95 lbs. =	79 „
	177 lbs.

$$M = 177 \times 18{\cdot}5^2 \times \frac{12}{8} = 91{,}000 \text{ lbs.-ins.}$$

$$d \text{ at } 120\,bd^2 = \sqrt{\frac{91{,}000}{120 \times 12}} = 7{\cdot}95 \text{ ins.}$$

Try an 8-in. tile with 2 in. cover = 10 ins. overall.

This will have a d of about 8·75 ins.

Make the ribs 5 ins. wide.

Actual weight, 2 ins. concrete =	25 lbs. per sq. ft.
One rib 5 ins. × 8 ins. at 17 ins. centres =	30 „ „ „ „
One tile 12 ins. × 12 ins. × 8 ins. at 17 ins. centres =	23 „ „ „ „
	78 „ „ „ „

This is very close to our estimated weight.

Flange thickness = 2 ins. concrete + ½-in. for tile = 2½ ins.

From Table HT I this flange resists 1250 lbs. per in. width

$$\frac{t}{d} = \frac{2{\cdot}5}{8{\cdot}75} = 0{\cdot}285$$

From Fig. 222 the lever arm is 0·88 d.

Maximum moment of resistance for f_c = 750 lbs. per sq. in.

$$= 1250 \times 0{\cdot}88 \times 8{\cdot}75 \times 12$$

$$= 115{,}000 \text{ lbs.-ins. (quite safe).}$$

$$A_s = \frac{91{,}000}{18{,}000 \times 0{\cdot}88 \times 8{\cdot}75} = 0{\cdot}655 \text{ sq. ins. per ft.}$$

$$\text{Area per rib} = 0{\cdot}655 \times \frac{17}{12} = 0{\cdot}93 \text{ sq. ins.}$$

We may use two bars $\frac{13}{16}$-in. or, if desirous of saving steel, one bar $\frac{3}{4}$-in. and one bar $\frac{13}{16}$-in. per rib. This use of different diameters would almost certainly lead to confusion on the job and might waste more money in time lost, than the money saved in weight of steel.

$$\text{Shear area per rib} = 5 \times 0{\cdot}88 \times 8{\cdot}75 = 38{\cdot}5 \text{ sq. ins.}$$

$$\text{Maximum shear per rib} = 177 \times \frac{18{\cdot}5}{2} \times \frac{17}{12} = 2320 \text{ lbs.}$$

$$\text{Maximum shear intensity} = \frac{2320}{38{\cdot}5} = 61 \text{ lbs. per sq. in.}$$

Care must be taken to distinguish between bending moment and shear per foot run of section and bending moment and shear per rib.

Floors Spanning Both Ways.—For these it is necessary to use tiles with closed-up ends as each tile is separated from its neighbours by concrete ribs which run in both directions. The best tiles are those specially made for the purpose but open-ended tiles may be used by closing up their ends with pieces of slate, sheet-metal or even timber. The bending moments will be somewhat larger than those given for solid slabs as the tiles are not so strong as solid concrete. To be on the safe side if a panel has a length l and a breadth b we may assume that part of the load is carried by the ribs in each direction say

$$\frac{w}{1 + \frac{l^4}{b^4}} \text{ carried by the ribs of length } l \text{ and}$$

$$\frac{w}{1 + \frac{b^4}{l^4}} \text{ carried by the ribs of length } b.$$

Quantities in Hollow Tile Floors.—The concrete may be estimated by taking the quantity required for a solid floor and subtracting the volume occupied by the tiles. The number of tiles per square yard will vary with the size and shape of the spans. Tiles which are 12 ins. long nominally, take up 12¼ ins. or 12½ ins. each when laid together and pointed with cement mortar. A span which will just take a whole number of tiles requires less concrete than one which is just too short as the spaces at the ends are made up solid. A foreman who lays out his tiles carefully will work in a greater number than one who lays them carelessly. The actual number of 12 ins. × 12 ins. tiles will be 6 to 8 per sq. yd. of floor. If the floor consists of a great number of similar bays it is possible to draw a plan showing the exact number required per bay. For a rough estimate we may assume 6 to 7 per sq. yd. for running out the concrete and 6·5 to 7·5 per sq. yd. when running out the number of tiles.

The steel when laid over steel beams as shown in Fig. 221 will have a length of about 1·30 × span giving a steel factor of 1·30 × 3·4 × 9 = 40 (lbs. per sq. yd.) for continuous bays. In using this factor care must be taken to choose the area per foot and not the area per rib. In the first example above we have one ¾-in. bar per rib = ·44 sq. ins. every 15 ins.

= ·352 sq. ins. every foot
= ·352 × 40 = 14·1 lbs. per sq. yd.

If the panels were not very long we could, by leaving out one bar at the end of each panel reduce this to 13 lbs. per sq. yd.

In the second example, which is non-continuous, using two bars

$$\tfrac{13}{16}\text{-in.} = 1{\cdot}04 \text{ sq. ins. every 17 ins.}$$
$$= 0{\cdot}735 \text{ sq. ins. every ft.}$$

we should have a steel factor of about 32

$$32 \times 0{\cdot}735 = \text{say 24 lbs. per sq. yd.}$$

Cross bars are not provided. For large and heavy spans stirrups may be necessary and even top bars to hang them on but this is exceptional. The centering is reckoned over the whole area, that part of the floor which requires only intermittent boards being separated and clearly marked. The difference in cost between a bay close-centered all over and one having boards at 15-in. centres is not great as both require the same supports.

Example.—Give the quantities of concrete and number of tiles per sq. yd. in the two examples worked out.

(*a*) 5½ ins. overall with 12 ins. × 12 ins. × 4 ins. tiles at 15-in. centres say 7 tiles per sq. yd.

$$\text{Concrete in solid } 5\tfrac{1}{2}\text{-in. floor} = 9 \times \frac{5{\cdot}5}{12} = 4{\cdot}13 \text{ cub. ft.}$$

$$\text{less 7 tiles 4 ins.} \times 12 \text{ ins.} \times 12 \text{ ins.} = \frac{7 \times 4}{12} = \underline{2{\cdot}34} \text{ ,, ,,}$$

$$1{\cdot}79 \text{ ,, ,,}$$

$$= 0{\cdot}066 \text{ cub. yds.}$$

Number of tiles to cover contingencies say 7·2 per sq. yd.

(*b*) 10 ins. overall with 12 ins. × 12 ins. × 8 ins. tiles at 17-in. centres say 6·0 tiles per sq. yd.

$$\text{Concrete in solid 10-in. floor} = \frac{9 \times 10}{12} = 7{\cdot}50 \text{ cub. ft.}$$

$$\text{less 6·0 tiles } 8'' \times 12'' \times 12'' = \frac{6{\cdot}0 \times 8}{12} = \underline{4{\cdot}00} \text{ ,, ,,}$$

$$3{\cdot}50 \text{ ,, ,,}$$

$$= 0{\cdot}129 \text{ cub. yd.}$$

Number of tiles to cover contingencies say 6·3 per sq. yd.

The concrete required to case the steel beams is kept separate and is often sub-divided into haunching, marked *h* on Fig. 221 and fire-proofing marked *f* on Fig. 221. [The author fails to see any necessity for this subdivision and an engineer running out his own quantities would, of course, put all the casing in one item.]

Hollow tile floors are mostly found with steel main beams but have been employed with reinforced concrete beams. Such main beams require steel top and bottom as a rule owing to the thin flange available for compression.

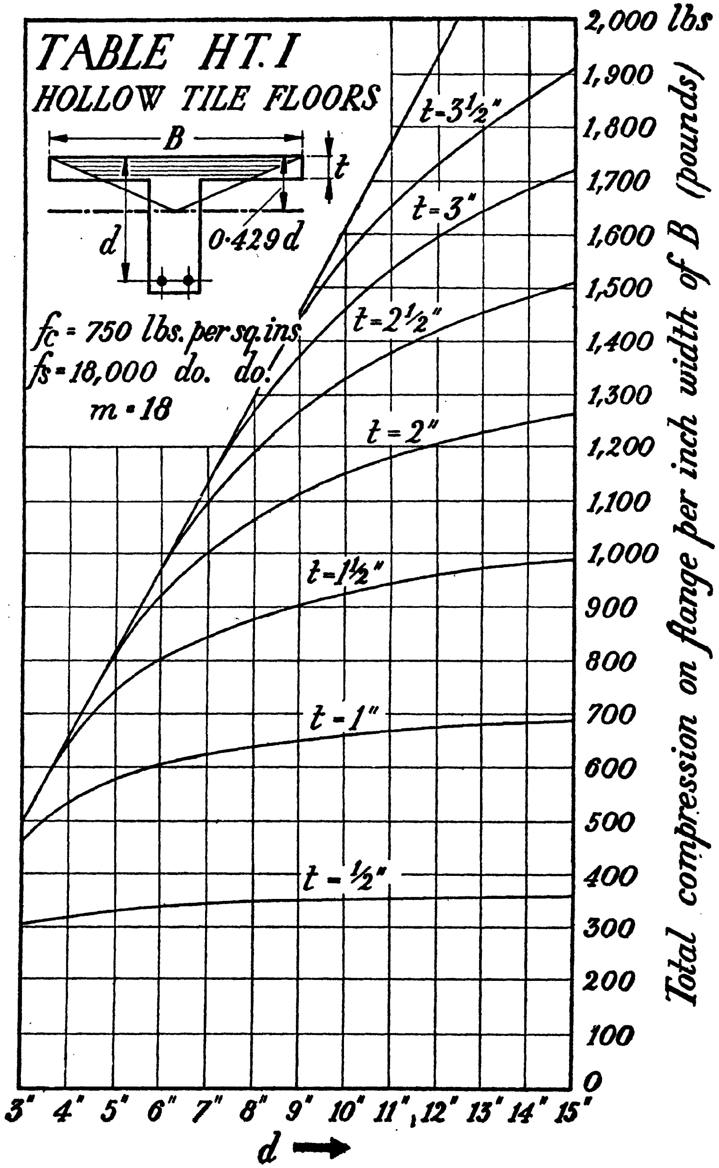


TABLE HT.I
HOLLOW TILE FLOORS
B
t
d
0·429d
f_c = 750 lbs. per sq. ins.
f_s = 18,000 do. do.
m = 18
t = 3½"
t = 3"
t = 2½"
t = 2"
t = 1½"
t = 1"
t = ½"
2,000 lbs
1,900
1,800
1,700
1,600
1,500
1,400
1,300
1,200
1,100
1,000
900
800
700
600
500
400
300
200
100
0
3" 4" 5" 6" 7" 8" 9" 10" 11" 12" 13" 14" 15"
d →
Total compression on flange per inch width of B (pounds)

CHAPTER XXIV

FLAT-SLAB FLOORS

Developed originally in America as a uniform slab supported on plain circular columns, the term flat-slab now includes several varieties of construction allied to the original design.

Varieties of Flat-Slab Construction.—Fig. 223 shows :—

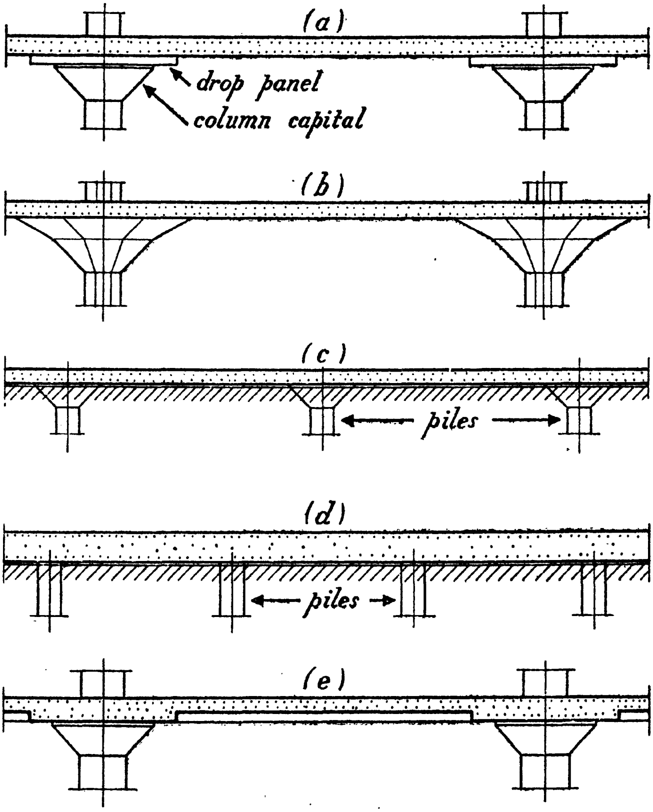


Fig. 223.

(a) Standard construction with conical column capitals and a square drop panel.

(b) A variation where a second slope on the capital replaces the drop panel.

(c) A type with capitals and no drop panels.

(d) A true flat-slab with heavy load and short spans.

(e) A sunk-panel flat-slab. The stiffest, most versatile and in the author's opinion the best type for all floor work.

General Method of Support.—The problem of the stresses in a continuous slab made of ideal elastic material cannot be solved for cases of general loading in the present state of our knowledge. A few simple cases have been solved but these involve long and difficult integrations and are beyond the scope of the practising engineer. Even if we could solve the general case of an elastic slab we should still be faced with the problem of applying the solution to a reinforced concrete floor. To give a very rough idea of how the floor supports itself we may imagine the floor cut into a series of strips in both directions. Fig. 224 shows a panel of standard flat-slab floor with the columns spaced at a distance L in both directions. The floor is divided by imaginary lines into strips marked A and B on the plan. Each strip is one half L wide. Strip A runs directly from column to column and may be regarded as the main beam. Section $x - x$ shows a longitudinal section of strip A under load. There is heavy local bending over the column head and fairly heavy positive moment half-way between the columns. Strip B is clear of the columns and may be regarded as a secondary beam supported by and spanning between the strips A. Section $y - y$ shows a longitudinal section through a strip B. The slab undulates gently up and down with small positive moments where two strips B intersect and small negative moments where strip B passes over strip A.

Sum of Bending Moments.—There is one particular relation between the total sum of the moments under symmetrical loading which is easily demonstrated. Fig. 225 is a plan of one square bay of flat-slab floor of span L supported on column capitals of diameter C. If this panel lies in the centre of many similar panels all carrying the same superload it is clear that each column capital carries the weight of one full panel of live and dead load, say W. By a simple equation, and neglecting positive and negative signs, the total moment across section $H M$, plus the total moment across section $B F$, plus the component of the moments on AB and FG resolved about the axis BF must equal the total load on the shaded half panel multiplied by the distance x between the centre of gravity of the shaded area and the centre of the shear forces on A B and F G. The total moment is very nearly

$$\frac{WL}{8}\left(1-\frac{2}{3}\frac{C}{L}\right)^2.$$

This total moment is seen to be the sum of the negative moment in one strip A at the support section, plus the positive moment in one

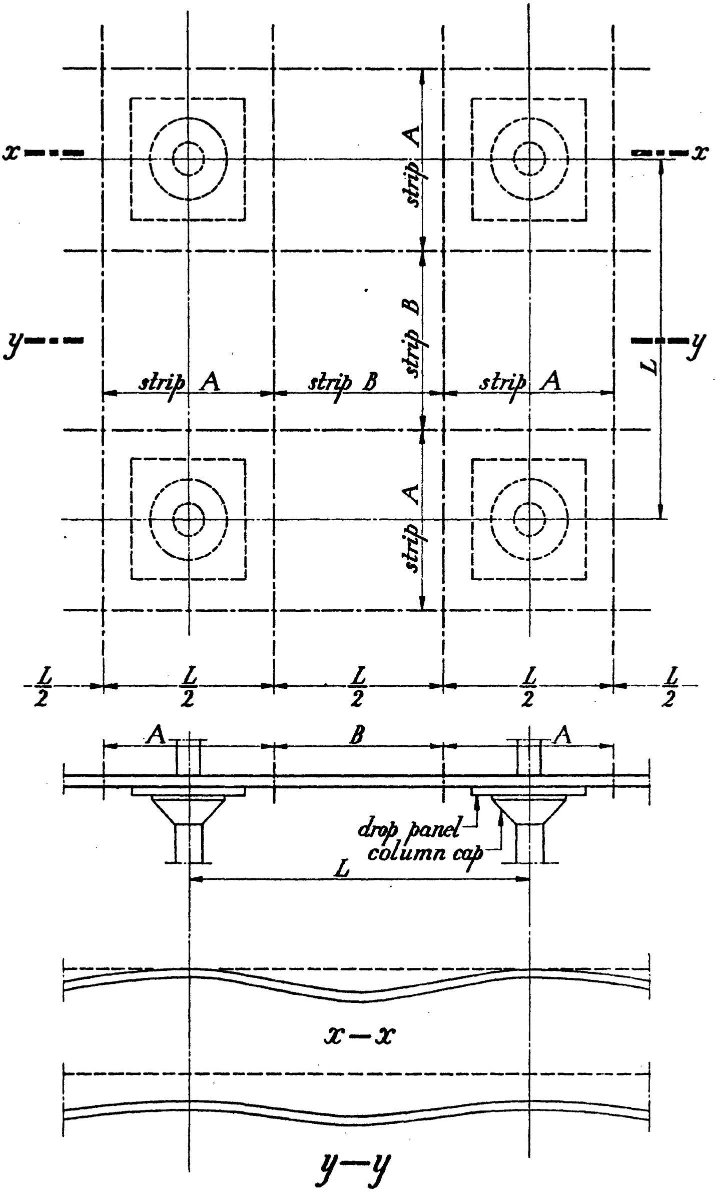

x
x
y
y
strip A
strip B
strip A
strip A
strip B
strip A
L
L/2
L/2
L/2
L/2
L/2
A
B
A
drop panel
column cap
L
x—x
y—y

Fig. 224.

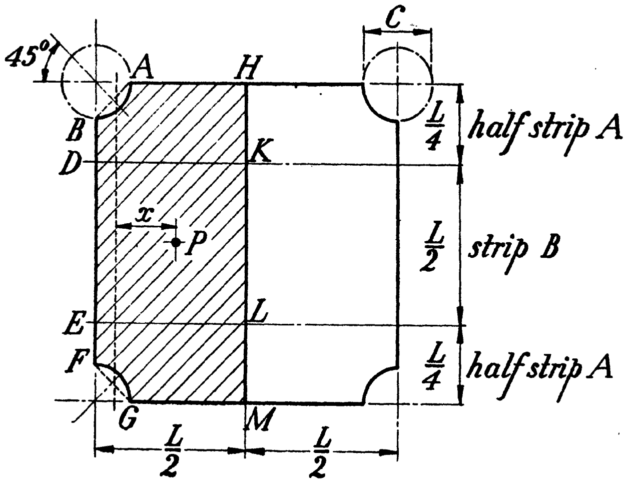


Fig. 225.

strip A at mid-span, plus the negative and positive moments in one strip B at " support " D E and at mid-span K L. If we take a standard column capital C equal to 0·225 L then

$$\frac{W\,L}{8}\left(1-\frac{2C}{3L}\right)^2=0{\cdot}725\,\frac{W\,L}{8}.$$

Theoretically this is only true for uniform loading over the whole panel and over all adjoining panels. If, however, we make the columns strong enough to take all the effects of unsymmetrical loading, we may rely on the Principle of Greatest Strength to redistribute the moments between the floor and columns according to the strengths provided.

Typical American regulations specify a total moment of only $0{\cdot}50\,\frac{W\,L}{8}$ to $0{\cdot}60\,\frac{W\,L}{8}$ for the sum of the moments in the floor sections but it is clear that these are only about 75 per cent. of the true values. The explanation of the discrepancy lies in the fact that the American moments are based largely on observed stresses in the steel reinforcement in actual floors under test load. It is clear that the tensile strength of the concrete in these tests has relieved the steel tension and given false values to the moments particularly at the sections where the moments are small. Unfortunately, this apparent strength is fictitious, as small deflections under test load do not guarantee that the structure is a long way from failure since a small further increase of load may serve to open tensile cracks at critical sections, throw all the tension on the steel and cause total failure at a load a little above twice the designed load. The high shrinkage of modern cements in this country (1935), particularly rapid-hardening cements, is sufficient to open cracks in floors designed for moments less than $0{\cdot}725\frac{W L}{8}$ and these shrinkage cracks destroy the power of the concrete to resist tension caused by bending moments. When designing thick floor slabs in this country, no allowance is made for any tensile strength of the concrete, and if we wish to design flat-slab work with the same factor of safety as slab-and-girder work, then the true moments must be taken and not the fictitious ones in typical American regulations. With a column capital, the diameter of which is $0{\cdot}225\ L$, the total of the positive and negative moments must not be less than $0{\cdot}725\frac{WL}{8}$, the exact value depending on the diameters and heights of the columns above and below the floor and the ratio of live to dead load. The intensity of moment will vary continuously along any cross-section of floor, but we can take average values for each cross-section of strip *A* and strip *B*. The distribution of the total moment cannot be calculated exactly, but it appears from the Principle of Greatest Strength that an approximately correct distribution will suffice if the total of the four moments is correct and the columns take the effects of unsymmetrical loading.

Tables of Moments, Shears, etc.—The following tables are based on the author's experience of designing and building flat-slab floors in this country. For comparison the Chicago Regulations (1918) are reproduced in the Appendix. They should be carefully read through as they explain many details of construction.

Table I.—Standard flat-slab floors with caps and drop panels. Columns above and below spaced *L* feet centres both ways. At least three panels wide and three panels long.

W_L = Total live load in pounds on one panel.
W = Total live and dead load in pounds on one panel.

Column Diameter.—Not less than twice thickness of strip *A* at mid-span, nor less than one-tenth of clear height under cap.

Minimum total slab thickness in inches $= \left(\frac{\sqrt{W}}{52} + 1\tfrac{1}{2}\text{ in.}\right)$, but not less than $\frac{L}{32}$, nor less than 6 ins.

Column Capital.—Not less than $0{\cdot}225\ L$ diameter measured where 1½ in. thick. Sides to slope at least 45° to the horizontal.

Drop Panel.—$0{\cdot}33\ L$ square: total thickness not less than $\left(\frac{\sqrt{W}}{35} + 1\tfrac{1}{2}\text{ in.}\right)$.

Shear Stresses.—Punching shear 120 lb. per sq. in. but 75 lb. per sq. in. beam shear on *b j d* area outside 45° line. (See Fig. 226.)

Moments—Interior Panels.—

$$\left.\begin{array}{ll}\text{Strip } A & +\dfrac{WL}{40} \text{ and } -\dfrac{WL}{28} \\ \text{Strip } B & +\dfrac{WL}{72} \text{ and } -\dfrac{WL}{72}\end{array}\right\} \text{Total } 0{\cdot}71\ \frac{WL}{8}.$$

The negative moment in strip A is to be taken on the section of the drop panel $\frac{L}{3}$ wide and $\left(\frac{\sqrt{W}}{35} \times 1\frac{1}{2}\text{ in.}\right)$ deep.

Moment in Interior Columns. $\frac{W_L L}{32}$ split between column sections above and below floor, i.e., about $\frac{W_L L}{64}$ on each.

Moments in External Panels if Outside Columns have Standard Caps and take Moment.

Increase positive moments 25%.

Moment in outside columns $\frac{WL}{30}$ split between columns above and below floor.

Moment in External Panels where Outside Columns take no Moment. Increase positive moments 50% and increase negative moments over first interior column 10%.

Stresses.—Concrete to be mixed 1 : 2 : 4, or richer, with a minimum test-cube strength of 2250 per sq. in. Stress not to exceed 750 lbs. per sq. in. on 1 : 2 : 4 mix, or one-third of minimum test-cube strength for richer mixture. The thicknesses prescribed for internal panels take care of the concrete stresses, but the stress in outside panels must be checked and compression steel used if necessary, or slab thickness increased.

Test Load.—Not to exceed twice live load plus dead load, and to be in position 48 hours. Deflections not to exceed $\frac{L}{800}$, nor to exceed calculated value by more than 25%.

Two-thirds of all deflections to be recovered 24 hours after load is removed.

NOTE.—With light super loads and external bays unstiffened by outside columns, the slab may need to be increased in depth to keep deflection below $\frac{L}{800}$.

Bressummers.—In addition to the weight of the outside walls, bressummers should be calculated to carry a load of $0{\cdot}2\ W$ to $0{\cdot}25\ W$, depending on the width of the outside columns.

Standard Flat-Slab Roofs.—While the floors are stiffened with columns above and below, the roof has columns underneath only.

On the other hand, for super loads of about 56 lbs. per sq. ft., the live load is smaller than the dead load and the two effects will tend to balance. If the roof is to carry a heavy live load the positive moment in strip *A* should be increased 20% above the value in Table I. In all cases the whole of the column bending moment must be taken on the column section below the roof.

Non-Standard Floors.—In cases where the column heights are excessive (e.g., reservoir roofs) it is inconvenient to use a standard column diameter of one-tenth the clear height. In cases, such as basement floors supported on isolated piles, little or no bending moment may be taken on each pile. (This, of course, does not apply to groups of piles.) In both these cases, the floors will clearly have to resist higher moments, since the columns supply no stiffening action. The columns may be given capitals to reduce the shear stresses and strengthen strip *A* over the support. The moments to be designed for are given in Tables II and III. For small spans the column capitals may be increased to 0·33 *L*, to act in the double capacity of caps and drop panels. Floors without caps or drop panels are usually adopted for foundation slabs of small span or basement floors cast on the ground.

Table II.—Flat-slab with standard column capital, but no stiffness in columns. Span *L* feet in both directions, and at least three panels wide and three panels long.

Minimum Total Slab Thickness—

Without drop panel $\left(\dfrac{\sqrt{W}}{39} + 1\tfrac{1}{2}''\right)$.

With drop panel $\left(\dfrac{\sqrt{W}}{45} + 1\tfrac{1}{2}''\right)$.

but not less than $\dfrac{L}{24}$, nor less than 6 ins.

Drop Panel—(if provided) 0·33 *L* square and

$$\left(\frac{\sqrt{W}}{31} + 1\tfrac{1}{2}''\text{ deep.}\right)$$

Moments in Interior Panels—

Strip *A* $-\dfrac{WL}{22} + \dfrac{WL}{30}$.

Strip B $+\dfrac{WL}{60} - \dfrac{WL}{60}$.

Moments in Exterior Panels and Next-to-End Support—

Strip *A* $-\dfrac{WL}{20} + \dfrac{WL}{22}$.

Strip B $+\dfrac{WL}{50} - \dfrac{WL}{50}$.

Table III.—Flat slab without capitals and no stiffness in columns. Span L feet in both directions and at least three panels wide and three panels long.

Minimum Total Slab Thickness—

$$\text{If no drop panel } \left(\frac{\sqrt{W}}{35} + 1\tfrac{1}{2}''\right).$$

$$\text{With drop panel } \left(\frac{\sqrt{W}}{40} + 1\tfrac{1}{2}''\right)$$

but not less than $\frac{L}{21}$, nor less than 6 ins.

Drop Panel (if provided) 0·33 L square and

$$\left(\frac{\sqrt{W}}{28} + 1\tfrac{1}{2}'' \text{ deep.}\right)$$

Moments in Interior Panels—

$$\text{Strip A } -\frac{W\,L}{18} + \frac{W\,L}{24}.$$

$$\text{Strip B } +\frac{W\,L}{60} - \frac{W\,L}{60}.$$

Moments in Exterior Panels and Next-to-End Supports—

$$\text{Strip A } -\frac{W\,L}{16} + \frac{W\,L}{20}.$$

$$\text{Strip B } +\frac{W\,L}{50} - \frac{W\,L}{50}.$$

Bending in the Columns.—The moments in the columns, where stiff columns are provided, are the sum of the moments in the section immediately above the floor and immediately below the cap. The moment should be divided between these two sections in the proportion of $\frac{I}{H}$. If the floor heights are equal and the column below the floor slightly larger than the column above, ⅜ of the moment may be taken on the column section above the floor and ⅝ on the section below.

Systems of Reinforcement.—In two-way systems there is a band of bars in strip A called the column-head band and a band of bars in strip B called the mid-band. It appears usual in American practice to assume points of contraflexure at the quarter points of the span and to stop or bend up all the bottom steel at these points. This is contrary to practice in this country. At least every third bar in strip A should be run through in the bottom about a foot past the column centre-line. In strip B every other bar may be run through. These precautions cover the shifting of the points of contraflexure under unequal loading and increase the shear strength. At points where there are two layers of bars the effective depth may be taken as the average effective depth of both layers. In four-way systems the mid-bands are replaced by short top bars to take the negative moment in strip *B*, and bands running diagonally from column to column giving four layers of bars over the column capital. For thick floors of standard construction a four-way system may be adopted.

Holes.—Holes in standard floors should be kept clear of strips *A*. Holes exceeding 2 ft. square or 0·1 *L* require additional steel each side of them, or stiffening beams in the floor, according to size. This is most particularly the case when designing to American regulations.

Sunk-Panel Floors.—The application of standard flat-slab construction is practically limited to cases where the outside columns have standard caps and take heavy fixing moments, as otherwise the moments and deflections in the outside panels are excessive. Then either the outside columns have to be made as wide as the column caps or the window heads kept down below the bottom of the cap. This can be avoided by bringing the outside columns into the building 3 or 4 ft. back from the outside walls, but, of course, they then detract largely from the clear floor space. Another practical limitation is the use of flat-slab work for superloads of 112 lb. per sq. ft. or less, as the thickness of the whole floor has to be $\frac{1}{32}$ of the span which, in a standard floor, will carry a superload of about 170 lb. per sq. ft. A further drawback is the uncertainty of the effect of leaving holes in the floors. It is quite clear that a hole 3 or 4 ft. square will cause big local reduction in stiffness and occasion heavy local stresses round its edges. Most regulations specify a minimum number of three complete panels of floor in both directions. This is frequently impossible in buildings having large internal areas or light-wells. Ordinary flat-slab work makes it very difficult to deal with wind stresses unless the thrusts can be taken on cross walls or gable frames.

All these drawbacks may be overcome by adopting a sunk-panel floor. In this form of construction, strip *A* appears as a shallow T-beam about 0·33 *L* wide, while strip *B* is a thinner slab giving the effect of a sunk panel in the ceiling. The extra depth of strip *A* allows the designer to dispense with stiff outside columns, to construct floors two panels or even one panel wide, to leave large holes in the thinner part of the floor and take wind bending stresses on the columns. It will be seen that the headroom measured under the projecting ribs is about the same as the headroom under the drop-panel of a standard floor.

Another strong point in favour of the sunk-panel type, is its greater available shear strength. Some of the main tension bars can be run through in the bottom of the projecting rib over the column capital. This in itself increases the shear strength, and in cases of heavy superloads (6 to 10 cwts. per sq. ft.) allows the use of vertical stirrups to give further shear strength. Coefficients for sunk-panel floors are given in the following Table IV :—

Table IV.—Sunk panel flat-slab floors, columns spaced L feet centres in both directions above and below the floor. At least three panels wide and three panels long.

Column Diameter as for standard flat-slab floor ; diameter not less than twice the thickness of strip A.

Slab Thickness.—Strip A to have projection 0·33 L wide and of a total thickness of $\left(\frac{\sqrt{W}}{35} + 1\frac{1}{2}''\right)$.

Thin part of slab 0·67 L square ; thickness $\left(\frac{\sqrt{W}}{63} + 1''\right)$.

Thick part of strip A to be not less than $\frac{L}{23}$ thick, and no slab less than 6 ins.

Column Capital.—As for standard flat-slab floor.

Moments and Stresses.—As for standard flat-slab floors.

Two Spans wide only with Stiff Outside Columns—

$$\text{Strip } A \quad + \frac{W\,L}{30} \text{ and } - \frac{W\,L}{25}.$$

$$\text{Strip } B \quad + \frac{W\,L}{60} - \frac{W\,L}{50}.$$

Moment in outside columns, $\frac{W\,L}{28}$, split between sections above and below floor.

One Span wide only with Stiff Outside Columns—

$$\text{Strip } A \quad + \frac{W\,L}{28}.$$

$$\text{Strip } B \quad + \frac{W\,L}{50}.$$

Moment in outside column, $\frac{W\,L}{26}$, split between sections above and below floor.

Rectangular Panels.—The length should not exceed 1·25 times the breadth. Strips A running in the direction of the length should be designed as strips A in square panels, the side of which equals the length. Strips A running in the direction of the breadth should be designed as strips A in square panels, the side of which equals the breadth. The positive and negative bending moments per ft. width of strip B should be taken as $\frac{W}{36}$ lb.-ft. in both directions, where W is the total load in pounds on the rectangular panel for internal panels and $\frac{W}{30}$ for external panels for floors with stiff columns, and $\frac{W}{30}$ and $\frac{W}{25}$ for floors where the columns have no stiffness.

Working Stresses.—Owing to the ease of construction and possibility of using a dry mix, a stress of one-third the test-cube strength may be used. With a 1 : 2 : 4 mix, 750 lb. per sq. in. should be obtained with reasonably good aggregate. A stress of 16,000 to 18,000 lb. per sq. in. on mild-steel bars is allowable. Punching shear and beam shear must be clearly distinguished. (Some continental designers will not recognize the condition of punching shear.) Full punching shear can only be allowed on the perimeter *a b c d* in Fig. 226.

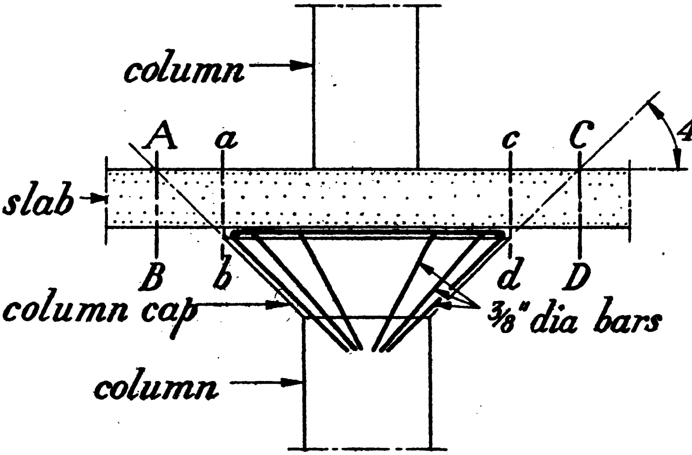


Fig. 226.

The perimeter *A B C D* must be strong enough in beam shear. With the thickness and sizes in Table I, the intensity of beam shear round the drop is $3{\cdot}3\sqrt{w}$ lb. per sq. in., where w is the inclusive load per sq. ft. If w exceeds 500 lb. per sq. ft. then the shear exceeds 75 lb. per sq. in., and either shear reinforcement must be provided (this is difficult in a thin slab) or the drop must be increased in size. Better still a sunk-panel design should be used.

Deflections.—It is notoriously difficult to forecast accurately what the deflection will be, but an approximate idea of the deflections of a standard floor loaded as shown in Fig. 227 may be obtained from Figs. 228 and 229. Fig. 228 gives the deflection of point X and Fig. 229 the added deflection of point Y below point X. In the figures w is the load per sq. ft. in pounds.

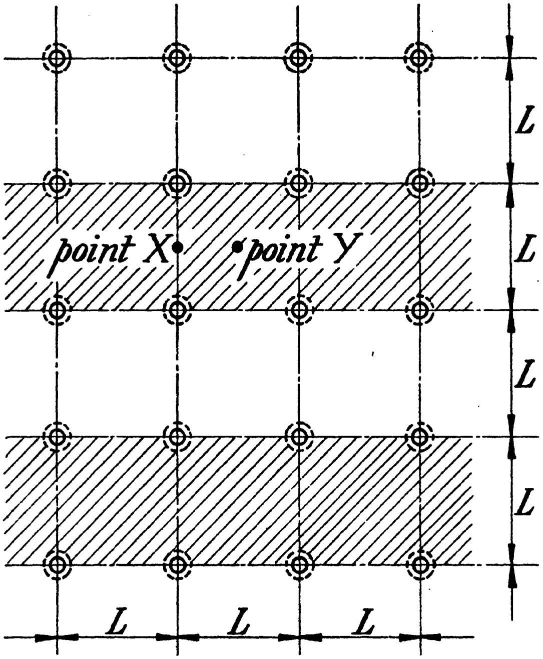


Fig. 227.

Example.—Design an internal panel of sunk panel flat-slab floor to carry 250 lbs. per sq. ft. with stresses of 750 and 18,000 ($m = 18$). The columns are spaced at 20 ft. 0 ins. centres in both directions.

The values are given in Table IV.

$$W = 20 \times 20\ (250 + \text{dead load per sq. ft.})\ \text{lbs.}$$

Assuming that the floor averages 10 ins. thick

$$W = 20 \times 20\ (250 + 120) = 148{,}000\ \text{lbs.}$$

$$\text{Thickness of strip } A = \frac{\sqrt{148{,}000}}{35} + 1\tfrac{1}{2}'' = 12\cdot5\ \text{ins.}$$

$$\text{Thickness of thin part of slab} = \frac{\sqrt{148{,}000}}{63} + 1 = 7\cdot1\ \text{ins.}$$

The area of the thin part of the slab is $0\cdot67^2$ or 45% of the panel area. If we make the floor 7 ins. and $12\frac{1}{2}$ ins. thick, the average thickness will be just on 10 ins.

Column capital diameter $= 0\cdot225 \times 20'\ 0'' = 4$ ft. 6 ins.

Column must be not less than $2 \times 12\frac{1}{2}'' = 25$ ins. diam.

Total punching shear strength around perimeter of cap

$$= 3\cdot14 \times 54'' \times 12\tfrac{1}{2}'' \times 120 = 255{,}000\ \text{lbs.}$$

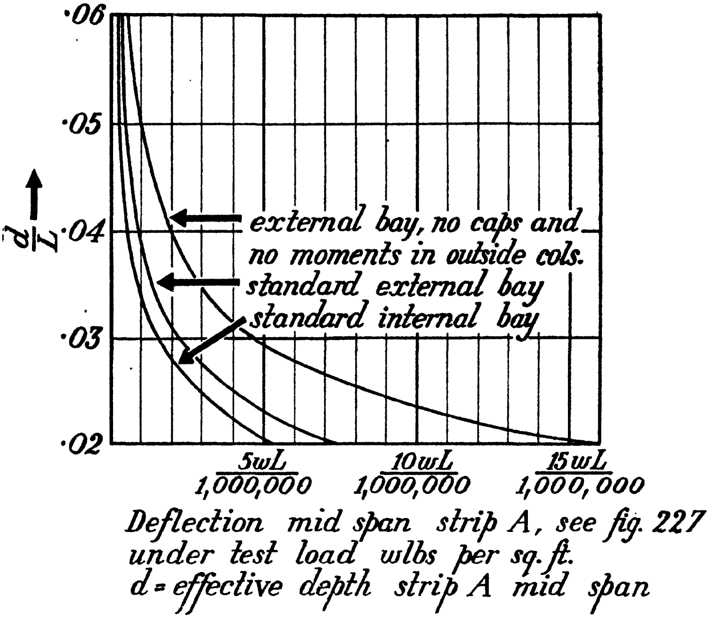


Fig. 228.

As this is nearly double the total value of W it is obviously safe. The diameter of the circle in which the 45° line cuts the top of the slab (see Fig. 226) is 79″ and its perimeter = 3·14 × 79″ = 248 ins.

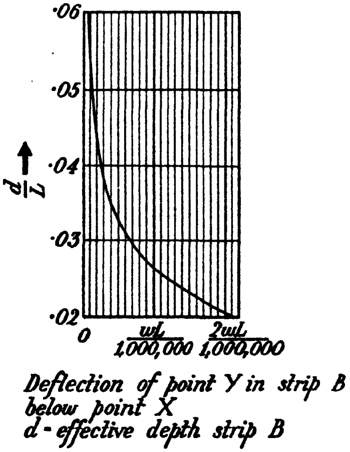


Fig. 229.

The shear a little further out is taken by four sections of the thick part of strip A. Each thick part of strip A is $0{\cdot}33L$ or 6 ft. 8 ins. wide and the width of the four sections is $4 \times 6' \, 8''$ or 320 ins. The total $b\,j\,d$ area on the perimeter of the 45° line is $248 \times j\,d$ or say

$$248 \times 0{\cdot}86 \times 11{\cdot}0'' = 2350 \text{ sq. ins.}$$

If $f = 75$, the total beam shear strength is

$$2350 \times 75 = 176{,}000 \text{ lbs.}$$

This also is more than W and is quite safe.

$$\text{Positive moment in strip } A = \frac{W\,L}{40} = \frac{148{,}000 \times 20}{40} \times 12$$

$$= 890{,}000 \text{ lbs.-ins.}$$

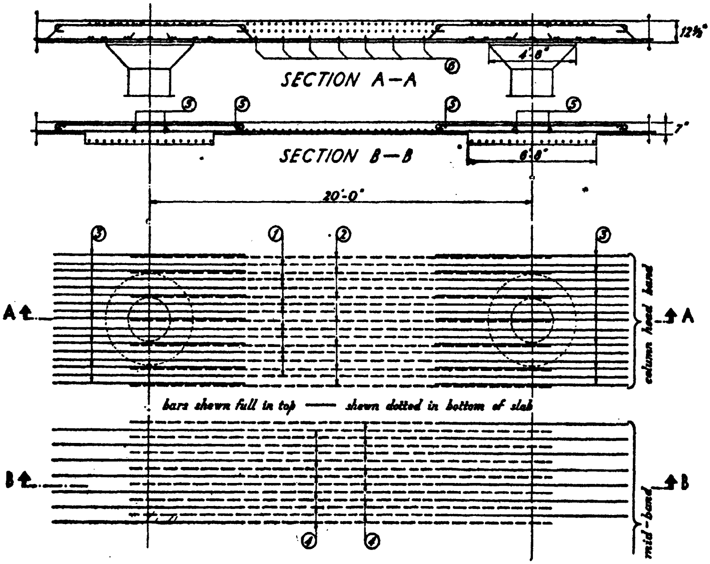
$$A_s = \frac{890{,}000}{18{,}000 \times {\cdot}86 \times 11{\cdot}25''} = 5{\cdot}06 \text{ sq. ins. say 17 bars } \tfrac{5}{8}\text{-in.}$$

$$\text{Negative moment in strip } A = \frac{W\,L}{28} = 1{,}270{,}000 \text{ lbs.-ins.}$$

$$A_s = \frac{1{,}270{,}000}{18{,}000 \times {\cdot}86 \times 11''} = 7{\cdot}45 \text{ sq. ins. say 25 bars } \tfrac{5}{8}\text{-in.}$$

$$\text{Positive moment in strip } B = \frac{W\,L}{72} = 492{,}000 \text{ lbs.-ins.}$$

$$A_s = \frac{492{,}000}{18{,}000 \times {\cdot}86 \times 6''} = 5{\cdot}3 \text{ sq. ins. say 27 bars } \tfrac{1}{2}\text{-in.}$$



PART PLAN shewing one column head band and half one mid-band

Fig. 230.

Using 17 bars in the column head band we should run at least 6 of these through in the bottom. In Fig. 230 an arrangement is shown where 7 bars are run through and 10 are bent up. The 10 bars bent up lap with 10 bars from the adjoining bay, giving 20 bars over the capital. We require 5 more bars. If we put a short top-bar over every straight bottom bar we shall have 27 bars in the top as shown in Fig. 230. The reinforcement in the thin part of the slab may be 27 bars all of one shape in each direction as shown. In addition we shall require four distributors to tie the top ends of the ½-in. bars and a few short distributors running across the bottom of the thick part of strip *A*.

For quantities it is best to run out the concrete and shuttering for one panel and take off the steel bar by bar.

Concrete	7″ slab 13′ 8″ × 13′ 8″ or 187 sq. ft.	= 4·03 cub. yds.
	12½″ slab (400 − 187) sq. ft.	= 8·20 ,, ,,
		12·23 ,, ,,
Shuttering	say 400 sq. ft. overall	= 44·5 sq. yds.
	plus 13′ 8″ × 5½″ × 4	= 2·8 ,, ,,
		47·3 ,, ,,

The different types of steel bar are numbered (1) to (6) on Fig. 230.

2 column head bands	= 20 bars (1)	⅝″ × 31′ 9″	= 660 lbs.
	+ 14 bars (2)	⅝″ × 23′ 0″	= 336 ,,
Extra bars over cap	= 14 bars (3)	⅝″ × 11′ 0″	= 161 ,,
2 mid-bands	= 54 bars (4)	½″ × 27′ 6″	= 990 ,,
Distributors	= 8 bars (5)	⅜″ × 11′ 0″	= 33 ,,
	14 bars (6)	⅜″ × 6′ 4″	= 33 ,,
			2223 lbs.

As there are 44·45 sq. yds. in one panel, the unit quantities per sq. yd. are 0·276 cub. yds., 50 lbs., and 1·06 sq. yds.

CHAPTER XXV

DOMES

Under Vertical Loads only.—A member made in the form of a thin shell whose shape is a surface of revolution, the axis of revolution being vertical, may be kept in equilibrium by a series of pure tensions and compressions under certain conditions. These conditions are :—

(1) That the surface is supported at a horizontal section.

(2) That all loading must be symmetrical about the axis of revolution.

[The fact that the member *may* be kept in equilibrium by direct stresses does not mean that it always *will* be so stressed under working loads. The equilibrium by direct stresses alone, represents only one possible method of support. It appears, however, from an extension of the Principle of Greatest Strength that the structure will be safe if we satisfy any one of the possible distributions of equilibrating stresses irrespective of all relative deflections.]

We will commence by taking a spherical dome under vertical loads as representing a common and fairly simple type. As such domes often carry ornaments at the top the dome will be assumed to carry a load W at the crown and its own self-weight, the surface being supposed of uniform thickness. Fig. 231 shows a spherical dome in vertical section through the centre line

Let W = point load at crown
w = self weight per unit area

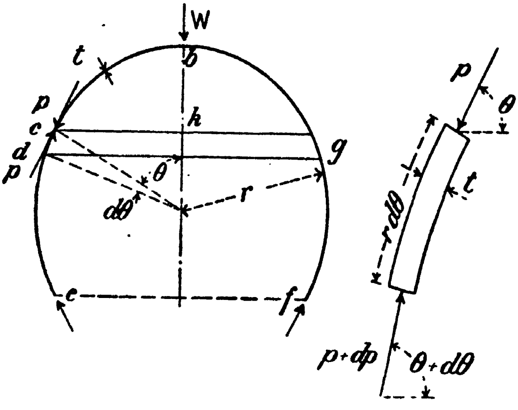


Fig. 231.

t = thickness (uniform) which is very small

r = radius

p = intensity of direct stress in the direction of the lines of longitude across the lines of latitude (taking W as the north pole).

s = intensity of hoop tension or compression along the lines of latitude (positive for tension). In the figure the left-hand view is a general section. The dome is supported along the circle $e\ f$, the direction of the supporting forces being, by our first assumption, tangential to the surface. Take an annular ring $d\ c\ k\ g$ bounded by two circles of latitude the width of the ring $d\ c$ being equal to $r\ d\ \theta$ where θ is the angle measured from the pole or crown. The stresses on this ring are

(I) a compressive stress p along its upper surface.
(II) a compressive stress $p + d\ p$ along its lower surface.
(III) its self-weight.
(IV) a hoop tension in the ring itself equal to s.

The horizontal radius of the ring is $k\ g = r \sin \theta$.

Taking the stresses on the upper surface first we see that the total force is $p \times$ area

$$= p \times t \times 2\ \pi\ r \sin \theta.$$

By our assumption this acts along the tangent directly up the lines of longitude at all points.

The total vertical component is :—

$$p\ t \times 2 \pi r \sin \theta \,.\, \sin \theta = 2 \pi\ r\ p\ t \sin^2 \theta.$$

This must equal the total weight above the section.

The weight of the dome itself = $w \times$ area

$$= w \times (b\ k) \times 2\ \pi\ r$$

$$= 2\ \pi\ r \times r\ (1 - \cos \theta)\ w.$$

Total weight

$$= 2\ \pi\ r^2\ (1 - \cos \theta)\ w + W = 2\ \pi\ r\ p\ t \sin^2 \theta \quad . \quad . \quad (1)$$

Turning to the right-hand view in the figure which shows an enlarged view of the section of this ring, we see that the forces on the upper surface tend to press outwards and cause hoop tension in the ring. The forces on the lower surface press inwards and tend to cause hoop compression.

The total stress per unit length of upper surface is $p \times t$.

The outward horizontal component is $p\ t \cos \theta$.

As the horizontal radius of the ring is $r \sin \theta$ this would cause, if acting alone, a hoop tension equal to

$$p\ t \cos \theta \times r \sin \theta.$$

The stresses on the lower surface would cause a hoop compression

$$(p + d\ p)\ t \cos (\theta + d\ \theta) \sin (\theta + d\ \theta).$$

The difference would be the total hoop tension which is

$$s \times (t\ r\ d\ \theta)$$

$$p\,t\cos\theta \times r\sin\theta = \frac{\cos\theta}{\sin\theta} \times \left(r^2(1-\cos\theta)\,w + \frac{W}{2\pi}\right)$$

by equation (1)

$$\therefore\ p\sin\theta\cos\theta = \frac{w\,r}{t}(\cot\theta - \operatorname{cosec}\theta + \sin\theta) + \frac{W}{2\pi r t}\cot\theta.$$

The increase of outward force due to an increase $d\,\theta$ which causes the hoop tension in our small annular ring is

$$-\frac{d}{d\theta}(p\,t\,r\sin\theta\cos\theta)\,d\,\theta = +\,s\,t\,r\,d\,\theta$$

$$\therefore\quad -s = \frac{1}{t\,r\,d\theta} \times \frac{d}{d\theta}(p\,t\,r\sin\theta\cos\theta) \times d\,\theta$$

$$= \frac{d}{d\theta}(p\sin\theta\cos\theta)$$

$$-s = \frac{d}{d\theta}\left(\frac{w\,r}{t}(\cot\theta - \operatorname{cosec}\theta + \sin\theta) + \frac{W}{2\pi r t}\cot\theta\right)$$

$$s = \frac{w\,r}{t}\,\frac{(1-\cos\theta-\cos^2\theta)}{(1+\cos\theta)} + \frac{W}{2\pi r t}\operatorname{cosec}^2\theta \quad . \quad . \quad . \quad . \quad (2)$$

In the particular case when $W = 0$

$$s = \frac{w\,r}{t}\,\frac{(1-\cos\theta-\cos^2\theta)}{(1+\cos\theta)}$$

$$\text{when } \theta = 0 \text{ at the crown } s = -\frac{w\,r}{2\,t}$$

$$s = 0 \text{ when } 1 - \cos\theta - \cos^2\theta = 0$$

$$\text{when } \cos\theta = \frac{-1 \pm \sqrt{5}}{2} = \cdot 618$$

$$\theta = 51^\circ - 48'.$$

Above this point the dome is under compression in both directions. (In the East a large number of domes are built of mud or allied material.) Below this point there is hoop tension, the top of the dome tending to collapse inwards and the lower part to burst outwards.

The addition of a point load W causes an additional tension

$$\frac{W}{2\pi r t}\operatorname{cosec}^2\theta.$$

This becomes infinite when $\theta = 0$ and the ornament will have to be distributed over a definite area by thickening up the top of the dome so that it is stiff enough to spread the load.

If W is sufficiently large, hoop tension may exist at all points. If the dome is a complete hemisphere it can be supported by vertical forces only and requires no horizontal thrust at the abutments the reason being that the hoop tensions in the dome itself counteract all tendency to outward thrust at all points of the height and supply horizontal thrust as and when required.

Application.—In a practical dome wind and temperature stresses have to be allowed for. In estimating W allow a large margin as architectural details have a habit of turning out larger than expected.

The designer will find that the calculated thickness is very small and the design is governed by practical considerations. Domes are not used as an engineering form of construction owing to the difficulty and cost of the centering. Having regard to the fact that the centering is curved in both directions a minimum thickness of 5 ins. seems indicated. Domes of 50 ft. 0 ins. diameter have been constructed with a thickness of 3 ins. only, but the author opines that the exactness of centering and difficulty of concreting would outweigh the material saved.

As an example suppose we have a dome 50 ft. 0 ins. diameter and 25 ft. 0 ins. high (i.e., a hemisphere) the dome being 5 ins. thick and weighing 63 lbs. per sq. ft.

$$p = \frac{wr(1 - \cos\theta)}{t\sin^2\theta} + \frac{W}{2\pi r t \sin^2\theta} .$$

$$= \frac{wr}{t(1 + \cos\theta)} + \frac{W}{2\pi r t \sin^2\theta} .$$

Suppose that $W = 10{,}000$ lbs. and is spread over a circle 5 ft. 0 ins. diameter $= \dfrac{10{,}000 \times 4}{\pi \times 5^2} = 512$ lbs. per sq. ft.

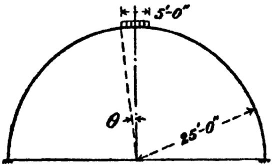


Fig. 232.

In Fig. 232 this extra load is shown to extend to $\theta = 5° 42'$.

Taking the heavily loaded portion separately as a small dome

$$w_1 = 63 + 512 = 575 \text{ lbs. per sq. ft.}$$

$$r = 25 \text{ ft. } 0 \text{ ins.}$$

$$t = \tfrac{5}{12} \text{ feet}$$

$$p = \frac{w_1 r}{t(1 + \cos\theta)} \text{ this increases as } \theta \text{ increases.}$$

When $\theta = 5° 42'$, $\cos\theta = \cdot 995$

$$p = \frac{575 \times 25 \times 12}{(1 + \cdot 995) \times 5} \text{ lbs. per sq. ft.}$$

$$= 17{,}300 \text{ lbs. per sq. ft.}$$

$$= 121 \text{ lbs. per sq. in.}$$

Hoop tension, considering the upper portion only, does not exist as θ never exceeds 5° 42′.

Taking the whole dome

$$p = \frac{wr}{t\,(1 + \cos\theta)} + \frac{W}{2\,\pi\,r\,t\sin^2\theta}$$

when $\theta = \frac{\pi}{2}$, $\cos\theta = 0$ and $\sin\theta = 1$

$$\begin{aligned} p &= \frac{63 \times 25 \times 12}{1 \times \cdot 5} + \frac{10{,}000 \times 12}{2 \times 3{\cdot}14 \times 25 \times 5 \times 1} \\ &= 3780 + 153 \\ &= 3933 \text{ lbs. per sq. ft.} \\ &= 27{\cdot}2 \text{ lbs. per sq. in.} \end{aligned}$$

Hoop tension

$$s = \frac{w\,r}{t}\,\frac{(1 - \cos\theta - \cos^2\theta)}{(1 + \cos\theta)} + \frac{W}{2\,\pi\,r\,t}\operatorname{cosec}^2\theta$$

$$\begin{aligned} \text{when } \theta = \frac{\pi}{2}, s &= \frac{63 \times 25 \times 12}{5} \times \frac{+1}{1} + \frac{10{,}000 \times 12 \times 1}{2 \times 3{\cdot}14 \times 25 \times 5} \\ &= 3933 \text{ lbs. per sq. ft.} \\ &= 27{\cdot}2 \text{ lbs. per sq. in.} \end{aligned}$$

$$\text{tension per foot of height} = \frac{3933 \times 5}{12} = 1630 \text{ lbs.}$$

Area of steel at 16,000 lbs. per sq. in. = ·1 sq. ins. It is difficult to calculate the wind and temperature stresses but it seems fairly evident that these cannot affect the safety of the dome having regard to the extremely small stresses caused by dead load. Such a dome would have not less than 14 to 16 lbs. of steel per square yard say ⅜-in. bars at 7 in. centres both ways. Half of these would be in rings round the lines of latitude and half in great circles up the lines of longitude both series being placed in the centre of the slab. The reaction on the supports would be 3933 lbs. per sq. foot or $\frac{5}{12} \times 3933$ = 1630 lbs. per ft. run.

Where openings occur in a dome they must be trimmed round. If the dome is very much cut up by large windows then the structure must be designed as a number of circular ribs the exact treatment depending on circumstances. Probably the best way of erecting large domes is to supply steel ribs which may be utilized to support the centering and case them in concrete.

The cone.—Let Fig. 233 represent a conical surface whose half angle is α.

Let W = point load

w = weight per sq. foot.

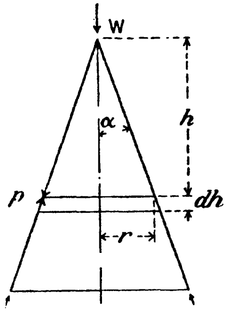


Fig. 233.

At any point distant h from the apex the radius is $h \tan \alpha$.

If p = pressure up slope
s = hoop tension
t = thickness as before

then

$$2\pi r p t \cos \alpha = W + 2\pi r \times \frac{h}{2} w$$

$$p = \frac{1}{2\pi r t \cos \alpha}\left(W + \frac{2\pi r h w}{2}\right).$$

The outward force causing hoop tension $= p\,t \sin \alpha$.

The height of the annular ring $= d\,h$, and slope length $= d\,h \times (\sec \alpha)$.

The hoop tension caused by the outward force is $r\,p\,t \sin \alpha$.

Area of section of ring $= t\,d\,h \sec \alpha$.

Total hoop tension in small ring $= s\,t\,d\,h \sec \alpha$.

Increase in total hoop tension in height $d\,h$ which is the total hoop tension in small ring

$$= \frac{d}{dh}(p\,t\,(\sin \alpha)\,r) \times d\,h$$

$$\therefore -s\,t\,(\sec \alpha)\,d\,h = \frac{d}{d\,h}(p\,t\,r\,(\sin \alpha))\,d\,h$$

$$-s = \frac{1}{t \sec \alpha} \cdot \frac{d}{d\,h}(p\,t\,(\sin \alpha)\,r)$$

$$= \sin \alpha \cos \alpha \frac{d}{d\,h}\left(\frac{W + \pi r h w}{2\pi t \cos \alpha}\right)$$

putting $r = h \tan \alpha$

$$-s = \frac{\sin \alpha \cos \alpha}{2\pi t \cos \alpha} \times \frac{d}{d\,h}(W + \pi h^2 w \tan \alpha)$$

$$s = -\frac{\sin^2 \alpha}{\cos \alpha} \times \frac{w\,h}{t}.$$

Thus the cone is always in compression in both directions under the loading shown. The reaction at the supports must be up the direction of the slope in accordance with our assumption of no transverse stress. The abutments therefore must resist a horizontal thrust of $p\ t \sin \alpha$ per ft. run. This may be met by putting a single tensile ring at the foot.

Extension to More General Loadings.—The principles underlying the two preceding discussions can be extended to other shapes and other loadings by an approximate method. Instead of choosing our annular ring of an infinitesimal height we can take a ring one foot or two feet high. We can always get the tension or compression acting up the lines of longitude by equating the total vertical component to the total weight coming on the structure up to that point.

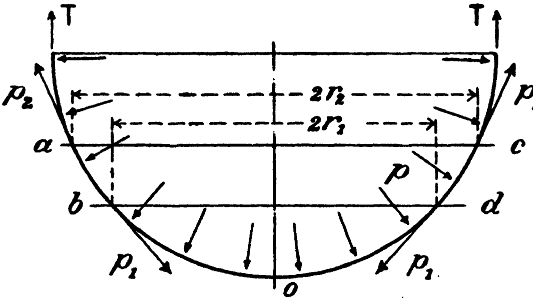


Fig. 234.

In Fig. 234 which shows the bottom of a tank or bunker suspended by tensions T, if we take two horizontal sections *a c* and *b d* and if p_1 and p_2 are the tensions up the lines of longitude then the vertical component of p_1 measured all round must equal the total vertical load on the bottom from *b* to *d*. Similarly the total vertical component of p_2 measured all round the circle *a c* must equal the vertical component of the load on the bottom from *a* to *c*. On the ring *a c b d* therefore we have the horizontal component of p_2 causing ring tension, the horizontal component of p_1 causing ring compression and the horizontal component of the internal pressure p causing ring tension. We can make the bottom any shape we like provided it is a surface of revolution and every horizontal section is a circle. By drawing out a large scale section we may choose the sections *a c* and *b d* fairly close together. The total loads, the values of r_1 and r_2, the inclinations of the tangents, the length of *a b*, etc., can all be scaled from the drawing. The author hopes that he has made the method clear and one example must suffice.

Example.—A water tank consists of a cylinder 20 ft. 0 ins. diameter with a hemispherical bottom suspended from the sides. The total maximum depth is 30 ft. 0 ins. What steel will be required in the bottom allowing a tension of 12,000 lbs. per sq. in.?

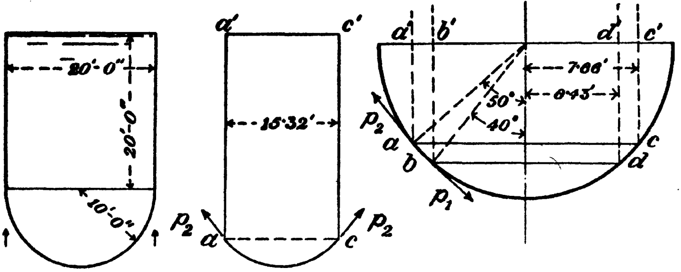


Fig. 235.

Take two sections at two horizontal planes *a c* and *b d* as shown in Fig. 235. We are going to consider the equilibrium of the ring *a b c d* cut off between these two planes. Choose *a* and *b* so that angles of 50° and 40° are subtended at the centre. Draw the cylindrical section *a* — *a'* and *c* — *c'*. The mutual pressure of the water outside the section and inside the section is normal to the surface of the cylinder and is always horizontal. Considering the equilibrium of this cylinder we have the weight of water enclosed, and a tensional force p_2 acting all round the circle *a c* inclined at 50° to the horizontal.

Let p_2 be the tension in lbs. per ft. run of section.

Perimeter of circle $= \pi \times 15{\cdot}32$ ft.

Total tension $= \pi \times 15{\cdot}32 \times p_2$ lbs.

Vertical component $= 15{\cdot}32\, \pi\, p_2 \times \sin 50°$.

This must equal the total weight of water.

The total weight may be obtained graphically by the Theorem of Pappus or, in this particular case, it may be taken from equation (*B*) which occurs later in this chapter and which applies to spherical cases only.

Volume of cylinder by equation (*B*)

$$= \frac{\pi \times 15{\cdot}32^2}{4} \times 20 + \frac{\pi \times 10^3}{3} \times 2\,[1 - \cos 50° + \cos 50° \sin^2 50°]$$

$$= 3710 + 1540 \text{ cub. ft.}$$

$$= 5250 \text{ cub. ft.}$$

Total weight $= 5250 \times 62{\cdot}5 = 329{,}000$ lbs.

Then $15{\cdot}32\, \pi\, p_2 \times {\cdot}766 = 329{,}000$

$$p_2 = \frac{329{,}000}{15{\cdot}32 \times \pi \times {\cdot}766} = 8900 \text{ lbs. per ft.}$$

The horizontal component of p_2 acting on the ring *a c b d* produces a ring tension.

Let s = intensity of hoop stress produced, in lbs. per ft. of section *a b*.

Then s (due to p_2 alone) $\times\ a\,b = 7{\cdot}66\,p_2 \cos 50°$

$$= \frac{7{\cdot}66 \times 329{,}000}{15{\cdot}32 \times \pi \times {\cdot}766} \times {\cdot}6428 = 43{,}800 \text{ lbs.}$$

Length of *a b* (scaled) = 1·75 feet

$$\therefore\ s = \frac{43{,}800}{1{\cdot}75} = 25{,}000 \text{ lbs. per ft. (tension)}$$

due to the action of p_2 alone.

Similarly by considering the cylinder $b\,b' - d\,d'$, we find p_1

$$p_1 \times 2\pi \times 6{\cdot}428 \cos 50° =$$

$$62{\cdot}5\left[\pi \times 6{\cdot}428^2 \times 20 + \frac{2 \times \pi \times 10^3}{3}[1 - \cos 40° + \cos 40° \sin^2 40°]\right]$$

$$2\pi p_1 \times 6{\cdot}428 \times {\cdot}6428 = 62{\cdot}5\,(2590 + 1150)$$

$$= 234{,}000 \text{ lbs., or } p_1 = 9050 \text{ lbs. per ft.}$$

Ring compression induced by p_1 in lbs. per ft. of section

$$= \frac{p_1 \cos 40° \times 6{\cdot}428}{1{\cdot}75}$$

$$= \frac{{\cdot}766 \times 6{\cdot}428}{1{\cdot}75} \times \frac{234{,}000}{2 \times 3{\cdot}14 \times 6{\cdot}428} \times \frac{1}{{\cdot}6428} = 25{,}500 \text{ lbs.}$$

We have still to consider the actual water pressure coming on the ring *a c b d*. Considering *a b* as a short straight line :—

Average depth from surface = 27·07 ft.
pressure per sq. ft. = 27·07 × 62·5
horizontal component = 27·07 × 62·5 sin 45° per sq. ft.
Total horizontal component = 27·07 × 62·5 cos 45° × 1·75 lbs. per ft. run of ring.

Average radius of ring = 7·07 ft.

Ring tension produced $= 27{\cdot}07 \times 62{\cdot}5 \times \frac{1}{\sqrt{2}} \times \frac{1{\cdot}75 \times 7{\cdot}07}{1{\cdot}75}$ lbs. per ft.

$$= \frac{27{\cdot}07 \times 62{\cdot}5}{1{\cdot}414} \times 7{\cdot}07 \text{ lbs. per ft. of section } a\,b$$

$$= 8500 \text{ lbs. per ft.}$$

Thus on the ring *a c b d* we have a series of forces p_2 which cause a ring tension of 25,000 lbs., together with a series of forces p_1 which cause a ring compression of 25,500 lbs. and a direct applied water pressure causing a ring tension of 8500 lbs.

The net result is a ring tension of

$$(25{,}000 + 8500 - 25{,}500) = 8000 \text{ lbs. per ft.}$$

At 12,000 lbs. per sq. in. this requires ·67 sq. ins. of steel per ft. of section *a b*, say ¾-in. bars at 8-in. centres.

In order to resist p_2 we shall require

$$\frac{8900}{12{,}000} = \cdot 74 \text{ sq. ins. say } \tfrac{3}{4}\text{-in. at 7-in. centres.}$$

To resist p_1 requires $\frac{9050}{12{,}000} = \cdot 752$ sq. ins. say ¾-in. bars at 6½-in. centres.

The steel therefore will consist of rings of ¾-in. bars at 8-in. centres running in a horizontal plane and ¾-in. bars running up the lines of longitude. Some of these may be stopped as they converge towards the south pole by being hooked round the rings (allowing for grip length).

By choosing angles of 10°, 20°, 30°, 60°, 70° and 80° the entire bottom may be calculated. This method is independent of the fact that the case we have taken happens to be a hemisphere as it may be applied to any shape, the angles, areas and radii being scaled off a drawing. As will be seen it is rather tedious.

Special Case. Water Towers with Spherical Bottoms.—All cases of tanks having hemispherical bottoms either suspended or dished upwards or bottoms consisting of segments of spheres may be solved by applying, with suitable modifications, the two theorems which follow.

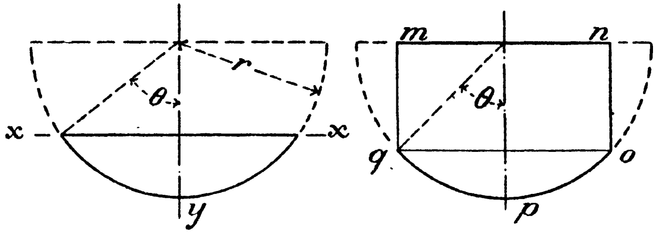


Fig. 236.

The volume of a sphere $x\,y\,x$ in Fig. 236 cut off by the plane $x - x$ is:

$$\frac{2\pi r^3}{3}\left[1 - \cos\theta - \frac{\cos\theta\sin^2\theta}{2}\right] \quad . \ . \ . \ . \ . \ . \quad \text{(A)}$$

The volume of a hemisphere cut off by the cylinder $m\,q - n\,o$ in Fig. 236 that is to say the volume $m\,n\,o\,p\,q$ is :—

$$\frac{2\pi r^3}{3}\,[1 - \cos\theta - \cos\theta\sin^2\theta] \quad . \ . \ . \ . \ . \ . \ . \quad \text{(B)}$$

Hemisphere Full of Water.—In Fig. 237 view I the thin hemisphere is just full of water up to the centre.

Let p = stress per unit run of section across the lines of latitude measured up the lines of longitude.

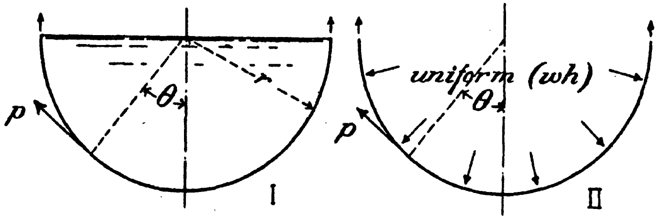


Fig. 237.

Let s = stress (ring stress) measured across the lines of longitude in the direction of the circles of latitude.

It is assumed that the hemisphere is reinforced by systems of bars to take these stresses.

$$2\pi r \sin\theta\,(p \sin\theta) = \frac{2}{3}\pi w r^3\,[1 - \cos\theta + \cos\theta \sin^2\theta]$$

where w = weight per unit volume of water

$$p = \frac{w\,r^2}{3}\left[\frac{1}{1 + \cos\theta} + \cos\theta\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hoop tension due to change in p

$$= \frac{d}{d\theta}(p\,r \sin\theta \cos\theta)\,d\,\theta \times \frac{1}{r\,d\,\theta}$$

$$= \frac{w\,r^2}{3}\left[\frac{3\cos^4\theta + 3\cos^3\theta - \cos^2\theta - \cos\theta - 1}{(1 + \cos\theta)}\right].$$

The ring tension due to direct pressure

$$= wr^2 \sin^2\theta \cos\theta.$$

Total ring tension = sum of these two.

$$s = \frac{w\,r^2}{3}\left[\frac{2\cos^2\theta + 2\cos\theta - 1}{(1 + \cos\theta)}\right] \quad . \quad . \quad . \quad (5)$$

and $s = 0$ when $\theta = 68° - 30'$.

Hemisphere under Uniform Internal Pressure.—Following a similar line of argument for case II in Fig. 237 we have

$$p = \frac{(w\,h)\,r}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$s = \frac{(w\,h)\,r}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

as might be seen from inspection.

All cases of water tanks with spherical bottoms may be solved by combining these two results in proper positive or negative proportions.

Example.—In the last example worked out, calculate p_2 and the ring tension by these formulae.

The pressure on the tank bottom in this case is composed of the weight of a hemisphere of water plus the weight of a cylinder of water 20 ft. 0 ins. high Fig. 235.

The stresses due to the former are given by equations (4) and (5) while the stresses due to the latter are given in equations (6) and (7).

For p_2, $\theta = 50°$

$$p_2 = \begin{cases} \dfrac{w\,r^2}{3}\left[\dfrac{1}{1+\cos\theta} + \cos\theta\right] & \ldots\ldots (4) \\ + \dfrac{(w\,h)\,r}{2} & \ldots\ldots (6) \end{cases}$$

$$\text{Ring tension} = \begin{cases} \dfrac{w\,r^2}{3}\left[\dfrac{2\cos^2\theta' + 2\cos\theta' - 1}{(1+\cos\theta')}\right] & \ldots\ldots (5) \\ + \dfrac{(w\,h)\,r}{2} & \ldots\ldots (7) \end{cases}$$

Where θ' = average of 40° and 50° = 45°
w = 62·5 lbs. per cub. ft.
r = 10 ft.
h = 20 ft.

$$\therefore\ p_2 = 62{\cdot}5 \times \frac{100}{3}\left[\frac{1}{1{\cdot}6428} + {\cdot}6428\right] + \frac{62{\cdot}5 \times 20 \times 10}{2}$$

$$= 2620. + 6250 = 8870 \text{ lbs. per ft.}$$

$$\text{Ring tension} = \frac{62{\cdot}5 \times 100}{3}\left[\frac{1 + 1{\cdot}41 - 1}{1{\cdot}707}\right] + \frac{62{\cdot}5 \times 20 \times 10}{2}$$

$$= 1720 + 6250 = 7970 \text{ lbs. per ft.}$$

Comparing these with the results worked out by the general graphical method we find as close an agreement as could be expected, the last method (which, however, applies to spheres only) being much quicker.

Domes—Conclusion.—Owing to difficulties of centering, domes are not popular forms of construction in England. The method given for spherical tank bottoms can be extended to spherical bottoms in bunkers, the hoop tension being reduced for higher values of θ according to the width of the bunker and the amount of arching to be expected in the coal. Cases which have not been specifically treated must be left to the reader as quite sufficient time has been spent on the subject here. In a practical case small complications due to self-weight will be met but these are, as a rule, easily allowed for.

CHAPTER XXVI

STAIRCASES

Although not constituting large structures, staircases are fairly common. Their design depends on the nature of the support available and is often complicated by somewhat indeterminate conditions.

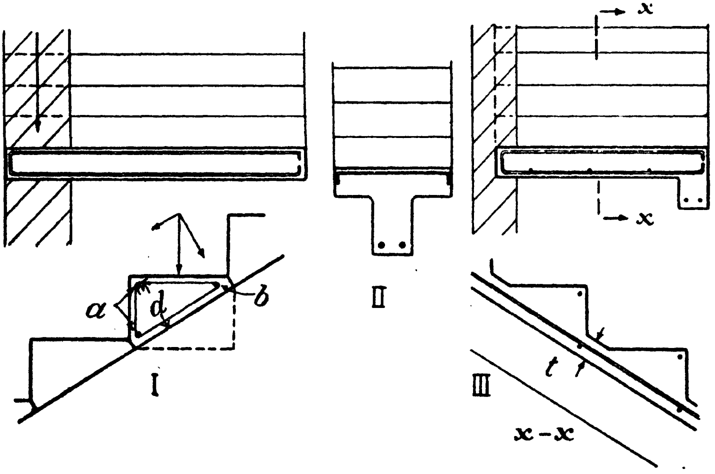


Fig. 238.

Horizontal Spans.—Three cases of stairs in which each step spans horizontally are shown in Fig. 238. In case I the steps are built into a brick wall and act as cantilevers. This case is only possible where the wall is thick enough and heavy enough to tail down the ends. The necessary thickness and height are usually laid down by local regulations. Such steps may very well be moulded separately beforehand, the tail which goes in the wall being square. They are then exactly similar to stone steps. The average thickness of concrete per square foot horizontal is about 4¾ ins., depending on rise and tread, and weight about 60 lbs. In designing them, the load, which is vertical, should be resolved along the principal axes of the section. Suppose the steps have a clear width of 4 ft. 0 ins. and carry a superload of 100 lbs. per sq. foot.

The maximum bending moment in a vertical plane is about

$$\frac{(100 + 60) \times 4^2 \times 12}{2} = 15{,}400 \text{ lbs.-ins.}$$

$$\text{say, by resolving} \quad 13{,}300 \text{ ,, ,,}$$

$$\text{and} \quad 8000 \text{ ,, ,,}$$

Assuming that the steel is arranged as shown, i.e., one bent bar *a* and one straight bar *b* per step with a few small links. Then the larger moment will be resisted by the upper leg of *a* acting at an effective depth *d* and the smaller moment will put bar *b* in tension. Taking *d* as about 5 ins.

$$\text{Area of bar } a = \frac{13{,}300}{18{,}000 \times 0{\cdot}857 \times 5} = 0{\cdot}172 \text{ sq. ins. say } \tfrac{1}{2}\text{-in. bar.}$$

The bar *b* may be ⅜-in. or ½-in. as most convenient.

Cantilever steps are not often found in reinforced concrete. In case II the flight is supported by a single beam under the centre. This is only suitable for very narrow flights as the twisting moment on the beam is dangerous. The stairs themselves cantilever over both sides of the beam and have very little bending moment. If the beam is to be calculated as a T-beam the minimum thickness of step at the back of the tread must be taken as the thickness of the flange. Case III shows steps supported on one side by a wall into which they are chased 4½ ins. and on the other side by a stringer. The steps span from wall to stringer the bending moment being small. The reinforcement may consist of a bent bar similar to *a* in case I say ⅜-in. diameter, with ¼-in. bars at 12-in. centres (or ⅜-in. bars at 18-in. centres) running down the flight to act as temperature bars. The thickness *t* is not less than 2 ins. and is obtained partly by introducing a fillet into the re-entrant angle. The stringer then becomes a ℸ-section with a 2-in. flange. As a variant of Case III two stringers may be used, one on each side. These may be reinforced concrete

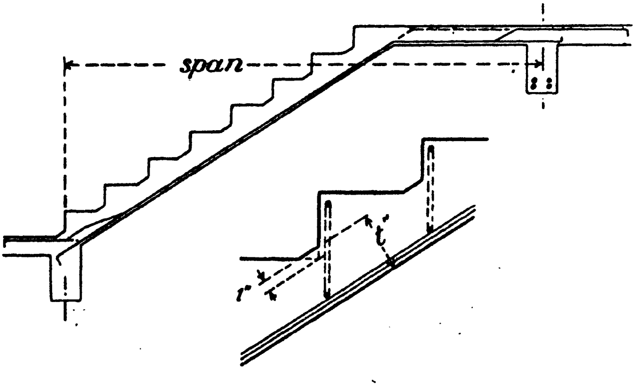


Fig. 239.

beams or steel joists. The employment of stringers in a staircase is always a nuisance both in design and construction and often looks unsightly. They should be avoided whenever possible by falling back on a flight span or mixed design.

Flight Spans.—Where the stairs consist of a straight flight unsupported at the sides they may be designed as a slab.

One such case is illustrated in Fig. 239. As there is nothing to induce end thrust the slab, although bent, will have the same moments as a straight beam spanning over the same horizontal distance. In the case illustrated we may take a moment of $\frac{wl^2}{10}$ at the centre of the span. In order to save concrete the thickness of the slab t ins. is increased by adding a fillet in the corner such fillet being 1 in. deep. If the steps are to be cased in thick marble a little deeper fillet say 1¼ ins. may be possible. The member then consists of a slab t ins. thick the projections taking no part in resisting the stresses. In estimating the weight per square foot horizontal, increase the thickness of t by 15% and add 3 ins. average thickness for the projecting steps.

Example.—A flight of stairs is unsupported at the edges and spans 9 ft. 0 ins. in plan being continuous both ends. It carries a superload of 120 lbs. per sq. ft.

Assume $t = 4\frac{1}{2}'' = \frac{4{\cdot}5 \times 150}{12} = 56$ lbs. per sq. ft. on slope

$= 56 \times 1{\cdot}15 = 65$ lbs. per sq. ft. on plan
add for projection 3″ $= 37$,, ,, ,, ,, ,, ,,
superload $= 120$,, ,, ,, ,, ,, ,,

222

$$M = 222 \times 9^2 \times \frac{12}{10} = 21{,}500 \text{ lbs.-ins.}$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{21{,}500}{12 \times 138}} = 3{\cdot}6'' \text{ say } t = 4\tfrac{1}{2} \text{ ins. and } d = 3{\cdot}75 \text{ ins.}$$

$$A_s = \frac{21{,}500}{18{,}000 \times 0{\cdot}857 \times 3{\cdot}75} = 0{\cdot}37 \text{ sq. ins.}$$

(use ½-in. bars at 6-in. centres = 0·39).

From first principles we should expect that the bend having tension on the inside would be the weakest spot as the steel would tend to pull out. In Fig. 239 the full lines show one arrangement of bars, alternate bars being bent up near the supports. Half these bars may be bent up at the corner as shown by the dotted line. The author knows of no reliable experiments which show exactly how strong such a bend is, and all thick slabs having large bars should be provided with ties at this point. The transverse steel may consist of short cross bars or one bent bar similar to the bar marked *a* in Fig. 238 provided in each step, the upper leg lying in the corner of the stair nosing.

Mixed Supports.—A large number of staircases are enclosed in brick walls as a precaution against fire. In such cases the outside edge of the stair is chased 4½ ins. into the surrounding wall.

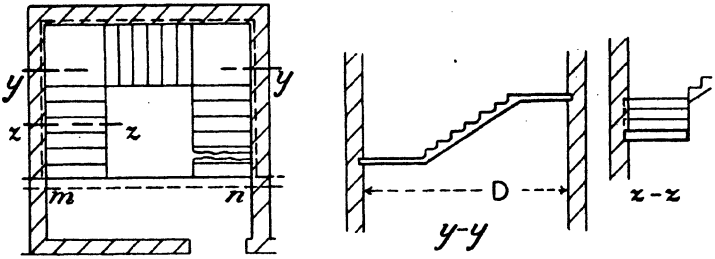


Fig. 240.

Fig. 240 shows a plan and section of such a case.

A large landing occurs at each floor level having one beam supporting it, marked *m* – *n* in the figure. The stairs, therefore, are supported at their extreme ends on the landing beams and, in addition, are supported along their outside edge by the wall. We may compare this with the case shown in Fig. 241 which shows a square slab having a square hole in the centre.

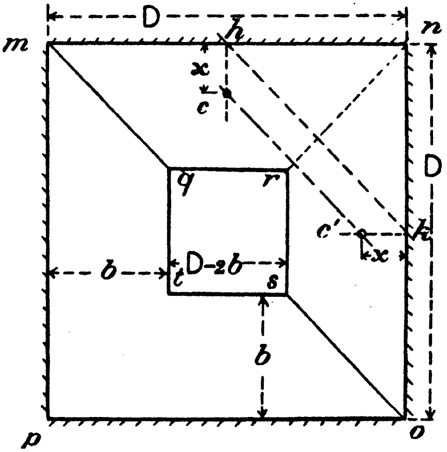


Fig. 241.

This slab is flat in plan and supported all round the edge, the reactions being either vertically up or vertically down, no bending stresses being imposed by the supports. Suppose the slab is uniformly loaded with w per square foot and consider the two sections $m\,q$ and $s\,o$. (The argument which follows is based on that in Morley's *Strength of Materials.*) Now whatever the distribution of the reactions along the sides it is evident from the symmetry of loading that the resultant of all the reaction forces on one side acts at the middle of that side. The resultant of all the reactions along the side $m\,n$ acts at h where h is the mid point, and this resultant reaction equals one-quarter of the total weight on the slab. The centre of gravity of the area of load $m\,q\,r\,n$ lies at point c a distance x from the side $m\,n$.

As $m\,q\,r\,n$ is a trapezium :—

$$x = \frac{b\,(D + 2\,(D - 2\,b))}{3\,(D + (D - 2\,b))} = \frac{b \times (3\,D - 4\,b)}{6\,(D - b)}.$$

Consider the equilibrium of the half $m\,q\,r\,s\,o\,n$.

The forces on it are

(1) A load equal to $\frac{1}{2} w\,(D^2 - (D - 2\,b)^2)$ acting at the centre of the line $c - c'$.

(2) The reactions of the sides $m\,n$ and $n\,o$ which may be replaced by two vertical forces acting at h and k each force

$$\frac{1}{4} w\,(D^2 - (D - 2\,b)^2).$$

(3) The stresses across the sections $m\,q$ and $s\,o$.

From symmetry it appears that the last item contains no shear reactions but consists entirely of bending moments.

Taking moments about the line $m\,o$.

Total moment on sections $m\,q$ and $s\,o$

$$= \frac{1}{2} w\,(D^2 - (D - 2\,b)^2) \times \frac{x}{\sqrt{2}}.$$

The length $m\,q$ plus $s\,o$ is $2\,b \times \sqrt{2}$.

Average bending moment per unit width of section

$$\doteq \frac{1}{2\sqrt{2}\,b} \times \frac{1}{2} w\,(D^2 - (D - 2\,b)^2) \times \frac{x}{\sqrt{2}}$$

$$= \frac{w\,b\,(3\,D - 4\,b)}{12}.$$

Writing $b = f\,D$ where f is a fraction.

$$\text{Average moment} = f\,\frac{(3 - 4\,f)}{12} \times w\,D^2.$$

This expression is a maximum when $f = \frac{3}{8}$ when it becomes

$$\frac{3}{64}\,w\,D^2 \text{ or } \frac{w\,D^2}{21{\cdot}3}.$$

If $f = \frac{1}{2}$, the average moment is $\frac{wl^2}{24}$ and we have the case of a slab without a hole. Now tests and experience in design show that $\frac{wl^2}{24}$ is the moment to be calculated for in such a case.

Arguing from analogy (a rather dangerous procedure where reinforced concrete is concerned) it appears that $f(3 - 4f)\frac{wD^2}{12}$ is the moment to be calculated for in the general case.

Returning to the case of an actual staircase we find many complications. The chasing of the edge into the wall may cause fixing moments. An actual thrust may exist down the inclined portions, being resisted by the landings and stairs themselves acting as girders in their own planes. (Compare the design of hopper bottoms.) Settlement of the brickwork would upset the distribution of stresses entirely. In practice the distance D in Fig. 240 usually lies between 9 ft. 0 ins. and 15 ft. 0 ins. and it is well known that slab thicknesses of 4 ins. to 6 ins. are sufficient to support such stairs. Some designers explain this by saying that the stairs, being chased into the wall, act partly as cantilever stairs, half the load being taken in this manner. The author cannot uphold this view as the stairs will stand up equally well when a cranked stringer is substituted for the wall. Other designers put forward the view that the load is largely transferred by direct thrust but cases do occur where this is scarcely possible. The author thinks that the strength of such stairs lies mostly in their square slab action as shown in Fig. 241. The reader should test this by procuring a piece of Bristol board and cutting it to the shape shown in Fig. 241. If he will compare the deflections produced when the slab is supported on all four edges (the corners being held down) with the deflections produced when the slab is supported on two edges only, he will see the nature of the stresses. Some years ago the author commenced a series of experiments on thin plates supported on three edges. The maximum curvatures, which occurred on the diagonals, were somewhat less than would be expected from the analysis above, although the two cases are not quite analogous. The plates showed a strong tendency to lift at the corners, the action at the two ends being a sort of torsional effect. Finally it would appear that, provided the quarter-space landings are fairly stiff, bending moments of $\frac{wD^2}{20}$ to $\frac{wD^2}{30}$ may be designed for, the smaller moment being taken for stairs which show a general continuity of shape. Thus for a stair having a width D of 14 ft. 0 ins. and carrying 120 lbs. per sq. ft. superload the moment would be say

$$(120 + 110) \times \frac{14^2}{20} \times 12 = 27{,}200 \text{ lbs.-ins.}$$

giving a 5-in. slab with $\frac{1}{2}$-in. bars at 6-in. centres.

The quarter-space landings should not be thinner than the slab in mid-span. The bars which run parallel to the wall and nearest to it are omitted only two thirds of the width of the flight being reinforced. All main bars may be kept at the bottom of the slab and must run all the way from wall to wall or from beam to wall. At the quarter-space landings the bars coming up from the flight below should be taken over the bars coming down from the flight above. The cross bars should consist of one bent bar as bar " *a* " in Fig. 238 per step either ¼-in. or ⅜-in. diameter.

Quantities in Stairs.—In a big job where the stairs are only a small item it is often extremely useful when running out a preliminary design and quantities, to have standard quantities which are near enough correct to be used without any further design. The reader should run out some of these for himself. For stairs supported all round, quantities of 0·18 cu. yds. and 15 lbs. per sq. yd. of plan area for $D = 10$ ft. 0 ins. in Fig. 240, to 0·23 cu. yds. and 24 lbs. per sq. yd. for $D = 15$ ft. 0 ins. may be used. The centering may be estimated at 2 sq. yds. per sq. yd. on plan. In a job where the staircases are a large percentage of the total quantities then they must be designed and more exact quantities taken. In balcony floors to theatres, etc., where very large steps occur the riser may consist of a vertical slab and the tread of a horizontal slab, or the floor may consist of a sloping slab having solid steps in coke-breeze concrete cast on afterwards.

In designing bent stringers to carry stairs which have outside stringers only, these stringers must be strong enough to carry a uniformly distributed load equal to the total load on the flight less the load on one quarter-space landing. No twisting effect need be allowed for.

In all the above it has been assumed that the slab is reinforced in tension only and worked to $R = 138\, b\, d^2$. This is the usual practice. If for some particular reason a very thin slab is required steel top and bottom may be used.

CHAPTER XXVII

DESIGN OF STRUCTURES

The difficulties encountered in this part of the subject consist chiefly in the choice of one line of action out of several alternatives. Every case which arises in practice has its own particular features and the circumstances which govern and limit the choice of one alternative or the other vary to an extraordinary degree. One type of structure may show slight advantages over another in use—a slight saving of headroom or floor space—and occasionally the appearance may come in as a minor deciding factor but, on the whole, it may be said that the choice is entirely guided by questions of cost. This being the case the relative costs of labour and materials is a point of prime importance. Unfortunately at the present time the absolute and relative cost of all items is an uncertain matter and the designer must make it his duty to keep in touch with the latest current costs.

Choice of Materials.—Timber is not employed for many permanent structures in England, although timber piles are fairly common and timber floors are used in small buildings. Steel and masonry construction are the two great competitors to be faced. It is impossible to exploit the design of reinforced concrete structures fully unless the defects and drawbacks of the material are known and realized. Much harm has been done in the past by attempting to apply reinforced concrete to types of structure for which it is not fitted. There are some enthusiastic devotees of the subject who appear to regard reinforced concrete as the panacea for all engineering difficulties.

It is quite true that there are many structures for which it is particularly suitable but there are others where alternative forms of construction yield better results. To quote one particular example we may mention single storey factory buildings. These are invariably constructed with steel stancheons and trusses and any attempt to compete with steel construction in this particular field will only lead to loss of time and energy. Compared with steel construction reinforced concrete is more bulky, takes longer to erect, requires more careful and more continuous supervision and when once completed it cannot be altered or added to. To take a case where headroom is essential, namely the main beams in a warehouse floor.

An 18 in. × 8 in. × 80 lbs. steel joist will resist a bending moment of about 2,500,000 lbs.-ins. and support a maximum shear of

100,000 lbs. A reinforced concrete T-beam 18 ins. × 8 ins. overall will take a moment of about 1,000,000 lbs.-ins. and support a shear of 28,000 lbs. If the size of the beam were determined by questions of shear alone, then a section 32 ins. × 16 ins. overall or 36 ins. × 14 ins. overall would be required to resist 100,000 lbs., although this size might be reduced in mid-span. When a steel structure is complete it can be thoroughly inspected in a few minutes. A glance will show whether the members are in their correct places and a few taps with a hammer will check the riveting. If the material has been inspected before leaving the shops there are very few points which can go wrong. A careful inspection of a reinforced concrete structure after completion will show very little, especially if the surfaces have been washed or rendered. It is impossible to tell if any particular bar has been correctly bent and placed or omitted altogether and, more important still, the quality of the concrete and the soundness of construction round the bars cannot be ascertained. Not only must the steel be inspected in position before concreting commences, but the actual process of concreting must be carefully watched.

The strength of an existing steel structure may be ascertained by measuring the sections of members: new beams can easily be cleated in on site. The strength of an existing reinforced concrete structure cannot be determined, except by partly demolishing the structure and assuming that the remaining part is similarly constructed. It is impossible to join a new beam into an existing beam or, at any rate, impossible to make a sound and reliable joint.

If, therefore, the engineer is called upon to design a structure where size of members or rapidity of construction are vital points or where supervision cannot be arranged or extensive alterations are to be expected he will adopt steel construction.

On the other hand if absolute bulk and weight are essential, masonry or mass concrete will be used. The first point to decide, therefore, is whether the structure is of a type where reinforced concrete construction can be adopted with advantage.

Pre-cast Systems.—Many attempts have been made to overcome some of the drawbacks to reinforced concrete construction by making parts of the structure in the workshop and erecting and assembling them ready-made and able to take load. These systems have not been generally adopted as no method can be found for supplying adequate shear strength at the joints. Thin concrete slabs may be pre-cast and placed on a light steel roof framework but the loss of T-beam strength and power to disperse local loads in heavier forms of construction, appears inseparable from the pre-cast system.

Balanced Structures.—There is one particular type of structure which should be treated with caution, namely the structure where the calculated loads balance one another leaving the structure free from stress. The basis on which all the loads are calculated must be examined and all possible variations allowed for. The most generally occurring cases are connected with the calculated pressures from earth backing. The active pressure from earth can never be relied on : it may act or it may not. Provision must be made to resist it in case it does act but no reliance can be placed on the fact that it will act. Counter-pressure from earth can always be relied on if the structure is free to move and follow the earth as it becomes compressed. In the case of a structure which is not free to move then neither the active pressure nor the counter pressure can be relied on.

Choice of Type.—It frequently happens that the designer may choose any one of a number of closely related types. (These types are discussed at some length in the chapters that follow.) Cost of construction is always the ruling factor but when in any doubt the simplest type should be chosen. Simple work is quick work and quick work is cheap work. Safeness and soundness of construction must be aimed at.

After all it is the owner who pays for the structure and the engineer's first duty is to satisfy the owner's requirements. There is a strong temptation to acquire kudos by designing some unusual or remarkable form of structure but this must be kept under.

Choice of Details.—Having chosen the type to be adopted we have then to choose the exact spacing of members.

This usually resolves itself into the question of whether a thicker slab spanning a greater distance is cheaper than a thinner slab and more beams. The answer will depend on the relative cost of concrete and centering. It is in this respect that simple and rapid methods of design are invaluable. A quick designer will be able to run out two or three alternative spacings and price them all roughly, choosing the cheapest, where a man who uses long and cumbrous methods will be able to deal with one set of spacings only.

Preliminary Designs.—In all competitive work a preliminary design is necessary. An owner requiring a structure will send out enquiries to four or five different firms who specialize in reinforced concrete. Some of these firms act both as designers and contractors while others confine themselves to design only and send out their designs and quantities to firms of contractors who send in tenders based on them. Thus the owners may receive ten or twenty different tenders, only one of which can be accepted. (In cases of well advertised jobs as many as one hundred and fifty different tenders have been received.) Allowing for all contingencies a fair average seems to be that one design out of every eight or every ten is proceeded with and the remainder are scrapped.

The whole object of a preliminary design is to arrive at the quantities and thence form a price in as short a time as possible. The main bending moments only are calculated and eighth-scale plans drawn. No attempt is made to draw out any working details. At the same time the designer must be clear in his own mind what types of members he intends to adopt and how he imagines the whole structure will be erected. A well-executed preliminary design is a great help if the design is accepted and details have to be prepared. In structures designed by a consulting engineer where only one design is prepared, a preliminary design of the whole structure should be made before any part is detailed as this allows of the adoption of standard sizes of beams, bars, etc., throughout the whole job.

Final Design.—Before commencing a final design it is as well to have the preliminary design checked. The designer should try to look at everything from the point of view of the contractor's foreman. Foremen as a whole are a highly efficient class and one for which the author has the greatest respect. They are not academic experts or gifted with extraordinary powers of observation or deduction. The designer himself is thoroughly conversant with the job and knows exactly how he intends all the details to go. A well thought-out scheme looks so simple and obvious to its creator that he is apt to imagine it will appear equally simple and equally obvious to others. This is an error. There are no two men who look at things in exactly the same light, or reason along exactly similar lines. The designer must, therefore, make his descriptive list and details as full as possible, adding notes and description at every possible point. A great many of these notes, etc., may appear as unnecessary or childish reiteration to the designer himself unless he is able, in imagination, to place himself in the position of the foreman carrying out the work. A foreman's main duty is to organize the labour on the job and procure quick and cheap construction. Most foremen are able to pick out details from poor or insufficient drawings but the time spent in doing so prevents the proper carrying out of their other duties. Always commence the drawings and descriptive lists in the order in which the work will be carried out.

Drawings.—It is almost impossible to make too many drawings when detailing a job for erection, and a preliminary estimate should never be allowed to go out without some sort of a drawing. Some designers imagine that the fewer drawings that are done the better. This necessitates writing long explanatory notes on such drawings as are made, such as " Note : beam (43) on Second Floor to be as beam (96) on First Floor but bars (106) to be $\frac{7}{8}$-in. diameter instead of 1 in. and bent up at 3 ft. 6 ins. from columns each end." A few moments reflection will show that such a course of procedure will cause endless delays on the job (and possibly mistakes). Time spent in drawing out extra details saves time in making the descriptive lists and obviates frantic enquiries from the job for enlightenment on obscure points.

Once the designer has the whole scheme clear in his mind, full details may be drawn out by a junior in surprisingly short time. Scanty drawings may appear to save a little time in the first instance but the author has invariably noticed that they lead to disaster in the long run. Drawings and calculations should be checked over by some person who has taken no part in the preliminary design. The reader is earnestly advised to study all detail drawings which appear in the technical press. Details prepared by an expert look simple and straightforward enough when finished and give no indication of the difficulties surmounted. If the reader will take the skeleton outline of some such structure and design and draw in the details for himself without referring to the details shown, he will not find it an easy matter.

Temperature and Shrinkage Stresses.—Expansion Joints.—The introduction of the modern type of Portland cement about 1923 has considerably altered the problem. Prior to that date long structures usually showed a number of fine hair-cracks (of the order 0·001 in.) after a few years of service. Similar structures built of modern cement show a lesser number of much wider cracks (0·01 in. to 0·1 in.).

It is clear that the stresses in many structures due to shrinkage are higher than the stresses due to the loading and how to deal with the high effective shrinkage of modern cements is the one problem in design to which there is at present no solution.

When using high-shrinking cement, the following lengths, although not eliminating shrinkage cracks, will make them controllable and the reader may work to them.

(1) Low trestle bridges, buildings, retaining walls, one expansion joint every 100 ft. 0 ins.

(2) Covered reservoirs, buried basements, one expansion joint every 200 ft. 0 ins.

(3) High trestle bridges or any structure on high flexible columns, one expansion joint every 300 ft. 0 ins.

(4) Long walls partly buried and partly above ground, one expansion joint every 50 ft. 0 ins.

Where expansion joints are constructed, a separate column or counterfort should be provided at each side of the joint, short cantilever panels being added. No attempt should be made to make one column serve by supporting the end of a long beam on a sliding joint on a bracket. There should be no stress of any kind across the joint.

Deflections.—It is very rarely that deflections have to be calculated in practice. Most specifications allow a deflection of one six-hundredth of the span. Well-designed structures under test load show deflections of about one-tenth this amount. The reader is referred to *Principles of Reinforced Concrete Construction* by Turneaure and Maurer for a discussion on the subject.

Calculations.—Calculations should be made as clear as possible consistent with brevity. Sketches drawn roughly to scale are a great help. It is quite easy to design a whole job on pieces of scrap paper writing down the sizes on the drawings and quantities as they are worked out. Calculations, however, must be made so that they are a permanent record of the loads and moments taken, available for reference or checking. They should preferably be written in ink on foolscap paper. In collecting up the calculations for a job, collect everything. All the rough work and all the cancelled sheets (these latter marked plainly " cancelled ") must be included and pinned together with a title sheet in front. The date, the office number of the job and the designer's initials should appear on every sheet. The superloads and stresses worked to may be noted if any doubt might arise.

CHAPTER XXVIII

BUILDINGS

Types.—In point of materials used and cost, these represent the most important structures met with. There are two main types of construction.

(1) Flat slab.

(2) Slab-and-girder.

Flat slab construction requires a fairly large rectangular plan and fairly close spacing of columns which must be the same, or very closely the same, in both directions (say 20 ft. 0 in. to 25 ft. 0 in. centres). It requires more concrete per square yard than slab-and-girder work and therefore weighs more but takes less steel and gives much simpler construction and centering. When comparing the price of the two types this saving in labour cost due to simpler centering must not be overlooked. If the plan is suitable flat-slab work should always be adopted. The only point to be decided is the spacing of the columns, for, when this is fixed, the details of the floor follow automatically.

The closer the spacing the cheaper the building. A large number of columns hampers floor space and looks ugly. The choice of spacing, therefore, is dictated by questions of utility and not by principles of design. In any case the spacing of columns must exceed the floor height as closer spacings give the whole building a peculiar appearance. Probably a spacing of 20 ft. 0 ins. both ways will give most general satisfaction.

Flat-slab work gives a greater clear headroom and this may be important. It also offers more resistance to fire but this is a point of no importance when compared with first cost unless a substantial reduction in fire insurance premiums can be effected. The general discussion of floor finishes and superloads given for slab-and-girder work will apply also to flat-slab work.

The great advantage of slab-and-girder construction is its adaptability to irregular floor plans or irregular spacing of columns. Owing to its greater complexity the subject of slab-and-girder floors will be treated in detail.

Slab-and-Girder.—Loads.—The load on a floor consists of three parts—the superload, the finishes and the self-weight. The first and second are unavoidable and will apply to any spacing of columns and beams. The first of these is a live load and may or may not function, while the second and third are dead and always in operation. In a great number of cases the superloads to be allowed for are laid down by local regulations and the designer has no choice in the matter. In buildings of the heavy warehouse type the superload is not very difficult to estimate and a uniform allowance for all members may be taken. In buildings of the factory type, where the floor carries light machinery, small areas may be heavily loaded. This means that the floor slab must be designed for a heavier superload than the secondary beams which in turn must be able to carry more load per square foot than the main beams. Such cases require care and the only reliable guide is to examine a factory similar to the proposed structure. If the author may be allowed to criticize the modern methods of calculating the loads on floors it appears that only buildings devoted to storage of materials should be designed for a constant value of so much per square foot for all members.

Office buildings, for example, may be designed for 100 lbs. per sq. ft. for floor slabs up to 10 ft. 0 ins. span, for 75 lbs. per sq. ft. for members carrying more than 50 sq. ft. of floor and less than 100 sq. ft., and for 50 lbs. per sq. ft. for members carrying more than 100 sq. ft. The provision of a fixed superload for all members leads to somewhat extravagant loads on the main beams in buildings of the domestic and office classes. In calculating the loads on columns it is usual to assume that the roof and top storey are fully loaded, that the next storey carries 90% of its live load, the next 80%, the next 70% and so on down to 50%. (Don't forget to make a small allowance for the weight of the column itself.) If the designer has already reduced the superloads on the main beams, as suggested above, reductions of 5%, 10%, 15%, etc., up to 50% may be taken. In buildings devoted to heavy storage on all floors no reduction can be allowed. Care should be exercised in designing book-rooms or libraries, as the loads encountered are surprisingly heavy. Allowances for partitions which may be put in afterwards should be made. If the weight of the partition per foot run in plan is more than half the superload per sq. ft., multiplied by the span of the slab in feet, the slab must be designed for an increased superload.

As all modern partitions are capable of spanning a horizontal distance which depends on the type of partition used, it must be left to the reader to make suitable allowance in all cases. A partition resting on a hollow-tile floor and running parallel to the ribs may be assumed as supported equally on two ribs while a partition running across the ribs may, as a rule, be neglected. All brick walls of 4½ ins. or upwards must be carried by means specially provided. For a list of suggested superloads the reader is referred to the Regulations issued by the London County Council.

The load due to finishes is not very large nor difficult to estimate. The indiscriminate insertion of wooden nailing fillets or " dots " in the top of the slab by the carpenters for the purpose of nailing down a timber floor must be suppressed and metal fastenings should be substituted wherever possible. If the concrete slab is called upon to furnish a wearing surface the upper half-inch or more may be composed of cement, sand and granite chippings, forming a granolithic finish. If laid at the same time as the rest of the slab it may be reckoned as part of the slab. If laid after several hours have elapsed it cannot.

The dead load due to the self-weight of the floor must be estimated before the members can be calculated and is usually estimated separately for each member. Some designers prefer to run out an average weight for the whole floor and use this for all members as an inclusive load. This means that the slab is called upon to carry its own self-weight and the average weight per square foot of the beams as well.

Choice of Spacings.—Decreasing the column spacings always decreases the cost. As the floor height is not likely to be less than 10 ft. 0 ins. for the lower stories a spacing of 12 ft. 0 ins. in one direction by, say, 16 ft. 0 ins. in the other may be taken as an absolute minimum. For a light superload (100 lbs. per sq. ft.) a spacing of 15 ft. 0 ins. × 25 ft. 0 ins. will give a good all-round floor and for a heavy superload a spacing of 14 ft. 0 ins. × 20 ft. 0 ins. The cheapest floor is the one having the thinnest slab, providing that the main beams are not deficient of compression at mid-span, and this latter fact should be roughly checked before starting. The designer should aim at keeping the total depth of construction inside 22 ins. or 24 ins. for a light floor and inside about 30 ins. for a heavy floor. If the purpose for which the place is intended needs larger spans then greater depths cannot be avoided. If long spans and shallow construction are essential then a steel frame building must be used owing to the weakness of reinforced concrete in shear. When competing with steel design, reinforced concrete firms are prone to attempt to cut their sections down to the depths used by steel designers. This generally results in weakness to shear stress and is not a principle to be followed. As indicating the limits to which slabs will provide compression flanges to main beams, widths of flange corresponding to twelve times the thickness, plus 8 ins., 10 ins. or 12 ins. may be taken. Take stresses of 750 and 18,000 ($m = 18$).

Slab = 4 in. B = 56 in.
4½ in. B = 62 in.
5 in. B = 70 in.
5½ in. B = 76 in.
6 in. B = 84 in.

Taking beams having overall depths of 18", 20", 22", 24", 26", 30", 36". After allowing two or three inches from the bottom edge to the centre of the steel and 1½ ins. to 2½ ins. depth from the top edge to the centre of compression we have approximate values of the effective depth and the lever arm. From Table TB I we have the compression value of the flange and can thus find the moment of resistance by multiplying this value by the lever arm.

Overall Depth =	18″	20″	22″	24″	26″	30″	36″
$t = 4''$	1,580,000	1,880,000	2,160,000	2,430,000	2,710,000	3,350,000	4,290,000
$t = 4\frac{1}{2}''$	1,840,000	2,190,000	2,540,000	2,880,000	3,250,000	4,030,000	5,210,000
$t = 5''$	2,130,000	2,570,000	3,000,000	3,440,000	3,880,000	4,850,000	6,320,000
$t = 5\frac{1}{2}''$	2,330,000	2,840,000	3,400,000	3,890,000	4,400,000	5,580,000	7,370,000
$t = 6''$	2,580,000	3,220,000	3,840,000	4,430,000	5,050,000	6,420,000	8,520,000

Fig. 242.

These values are tabulated in Fig. 242 and show approximately the maximum moments which a T-beam will resist without compression steel. These moments may be increased by increasing the width of the beam rib, but are near enough to act as a guide in choosing floor thicknesses.

Using a 4-in. slab with main beams 24 ins. overall a moment of resistance of about 2,430,000 lbs.-ins. can be raised at mid-span without using compression steel. Now for the purpose of choosing a thickness we may assume that the load on the main beam is uniformly distributed and equal to the inclusive load on the floor multiplied by the span of the secondary beams. An example will make this clear. Suppose the columns are spaced 16 ft. 0 ins. one way and 24 ft. 0 ins. the other way, the superload being 224 lbs. per sq. ft.

$$\begin{array}{rll} \text{Inclusive load} = & 224 \text{ lbs. per sq. ft. superload} \\ + \text{ dead load estimated} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} 63 \text{ ,, ,, ,, ,, slab} \\ 16 \text{ ,, ,, ,, ,, secondaries} \\ 14 \text{ ,, ,, ,, ,, main beams} \end{array} \\ & \overline{} \\ \text{Say with finishes} & 324 \text{ ,, ,, ,, ,,} \end{array}$$

The load per foot run on the main beam is 324 × 16 = 5180 lbs.

$$\text{Bending moment} = 5180 \times 24^2 \times \frac{12}{12} = 3{,}000{,}000 \text{ lbs.-ins.}$$

Shear = 5180 × 12 = 62,000 lbs.

Shear area required is 275 sq. ins.

This shear area is always too high as one secondary beam always frames direct into the column, thus carrying part of the load direct. Referring to Fig. 242 we see that a 5-in. slab with a main beam 22 ins. deep will resist the bending moment without compression steel and a $4\frac{1}{2}$-in. slab with a main beam 24 ins. deep will resist the bending moment with a little compression steel.

If headroom is of no importance a 4-in. slab and a main beam 28 ins. overall would suffice. If headroom is very important a 6-in. slab with a beam 18 ins. deep would be suitable. The procedure then, in deciding the details of a floor is as follows. Choose the column spacing as close as can be reasonably managed making the end spans smaller than the centre spans if possible. Fix on a maximum allowable depth of construction. Rough out the bending moment on the main beams and choose a slab thickness that will resist the moment without much compression steel, then divide the panels into even spaces making the slab span as far as possible. The author thinks it is always as well to aim at eliminating this compression steel, as a large amount is required materially to increase the strength of the beam and this large amount tends to sever the rib of the beam from the flange and hinders concreting. If one bay of main beams is very much longer span than the others it may be advisable to use a thicker slab in the centre of this bay, spacing out the secondaries somewhat. For buildings having a very irregular plan an average value must be chosen. In locating the columns for such a building it is as well to start with the wall columns, placing a column at each angle in the wall.

Drawings.—When designing a floor an eighth-scale drawing is essential and all columns and beams should be lettered or numbered in some system. Any system will do and the designer should follow the one in vogue in the particular office where he is working. Some firms draw a plan of the floor as seen from above. On this plan all the projecting ribs of the T-beams, being underneath the floor, are indicated by dotted lines. Some firms draw a similar plan but show the beams in full lines. The author prefers the latter method, both for quickness and appearance. Although the method seems illogical mistakes do not occur. All floor plans should show what superload has been designed for and preferably should indicate the reactions of one beam on another in thousands of pounds.

Quantities.—When running out quantities from a preliminary design all the mid-span sections are considered. (In bridge design where the rolling load causes heavy shears at all points a closer examination is necessary.) The sections at the supports are only examined for total shear. No effort is made to design these sections in bending, for a preliminary design, and the exact arrangement of shear steel is not gone into. It is necessary, therefore, to employ some form of steel factor for the beams which shall include all laps, etc., and all stirrups. (This is not necessary in bridge work where the arrangement of bars and spacing of stirrups should be looked into from the very first.)

Suitable factors are given in Chapter XIV.

The reader should see what factors are in use in his own particular office. In using the factors given, it is necessary to arrange the main steel so that it will take nearly all the shear not taken by the concrete and this can generally be managed without much trouble.

Calculations.—The preliminary calculations for one bay of floor should be put on one sheet.

Example.—A floor is to be designed to carry 200 lbs. per sq. ft. on the slab, 150 lbs. per sq. ft. on the secondaries and 100 lbs. per sq. ft. on the main beams: all superloads. The columns are spaced 17 ft. 0 ins. × 28 ft. 0 ins. for the internal spans. Design one internal bay of this floor for stresses of 750 and 18,000. Weight of floor finishes is 10 lbs. per sq. ft. The total overall depth of main beams should not exceed 26 ins. or 27 ins.

[Average weight on main beams roughly

$$(100 + 10 + 80) = 190 \text{ lbs. per sq. ft.}$$

$$M = (17 \times 190) \times 28^2 \times \frac{12}{12} = 2{,}550{,}000 \text{ lbs.-ins.}$$

From Fig. 242 we can probably manage with a 4-in. slab and a main beam 26 ins. deep overall. The slab carries (200 + 10 + 50) lbs. per sq. ft. = 260. At 138 bd^2 with $d = 3{\cdot}25$ ins., a 4-in. slab has a moment of resistance of 17,600 lbs.-ins. Its maximum span as an interior bay is l where $260 \times l^2 \times \frac{12}{12} = 17{,}600$. This gives $l = 8{\cdot}24$ ft. As the bays are 28 ft. 0 ins. long we shall divide each into 4 giving a span of 7 ft. 0 ins. All this preliminary work will be done mentally and by slide-rule and need not be written down.]

The designer will make a rough sketch of the bay for his own information. This is shown in Fig. 243.

Slab span 7 ft. 0 ins. Load = 200 lbs. per sq. ft. super.
10 ,, finish
50 ,, self-weight

260

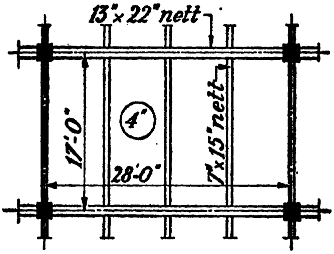


Fig. 243

$$M = 260 \times 7^2 \times \frac{12}{12} = 12{,}700 \text{ lbs.-ins.}$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{12{,}700}{138 \times 12}} = 2{\cdot}77 \text{ ins. use 4-in. slab with } d = 3{\cdot}25 \text{ ins.}$$

$$A_s = \frac{12{,}700}{18{,}000 \times 0{\cdot}857 \times 3{\cdot}25} = 0{\cdot}253 \text{ sq. ins. use } \tfrac{3}{8}\text{-in. every 5 ins.}$$

$$= 0{\cdot}26.$$

Quantities per sq. yd.	Concrete	0·11 cu. yds.
	Steel $(0{\cdot}26 \times 40) + 3$ say	13 lbs.
	Centering	1·0 sq. yds.

Secondary span 17 ft. 0 ins.

$$\begin{aligned} \text{Load} = 7 \times (150 + 50 + 10) &= 1470 \text{ lbs. per ft.} \\ \text{estimated self-weight} &= 100 \;,, \;\; ,, \;\; ,, \\ & \overline{1570} \;,, \;\; ,, \;\; ,, \end{aligned}$$

$$M = 1570 \times 17^2 \times \frac{12}{12} = 455{,}000 \text{ lbs.-ins.}$$

$$\text{Shear} = 1570 \times \frac{17}{2} = 13{,}400 \text{ lbs., at } 225 \text{ lbs. per sq. in.} = 60 \text{ sq. ins.}$$

We could use a beam with 2 heavy bars as Fig. 25, but a beam as Fig. 22 with 4 smaller bars is better.

If we use 4 bars ¾-in. (1·76 sq. ins.) a beam with a lever arm of 14·4 ins. will do (7 ins. × 15 ins. projection). If we use 4 bars ⅞-in. (2·40 sq. ins.) a beam 8 ins. × 15 ins. overall will do.

Using the former we have

$$d = 17 \text{ in., lever arm} = \text{say } 15 \text{ ins.}$$

$$\text{Total compression} = \frac{455{,}000}{15} = 30{,}300 \text{ lbs.}$$

Value of 4-in. flange = 2180 lbs. per inch (Table TB I)

necessary width $B = 13{\cdot}9$ ins. only.

$$A_s = \frac{30{,}300}{18{,}000} = 1{\cdot}68 \text{ sq. ins. use four bars } \tfrac{3}{4}\text{-in.} = 1{\cdot}76 \text{ sq. ins.}$$

Quantities per yd. run	Concrete $7 \times 15 \times \frac{1}{36^2}$	0·081 cu. yds.
	Steel $18 \times 1{\cdot}76$	32 lbs.
	Centering $\frac{7 + 15 + 15}{36}$	1·03 sq. yds.

Main Beam span 28 ft. 0 ins.

$$\begin{aligned} \text{Load} = 17 \times (100 + 50 + 10) &= 2720 \text{ lbs. per ft.} \\ \text{average of secondaries } \frac{110 \text{ lbs.} \times 17}{7} &= 267 \;,, \;\; ,, \;\; ,, \\ \text{estimated self-weight} &= 220 \;,, \;\; ,, \;\; ,, \\ & \overline{3207} \;,, \;\; ,, \;\; ,, \end{aligned}$$

$$M = 3207 \times 28^2 \times \frac{12}{12} = 2{,}530{,}000 \text{ lbs.-ins.}$$

Shear = 3207 × 14 less the reaction from one secondary that comes directly on the column.

$$= (3207 \times 14) - 10{,}400 = 34{,}800 \text{ lbs.}$$

$$\text{Min}^{\text{m}}.\ \text{shear area at 225 lbs. per sq. in.} = \frac{34{,}800}{225} = 155 \text{ sq. ins.}$$

Using a beam 26 ins. deep overall with 3 layers of bars we have d = about 22½ ins. and lever arm = 20½ ins.

$$\text{Total compression} = \frac{2{,}540{,}000}{20{\cdot}5} = 123{,}000 \text{ lbs.}$$

Value of 4-in. flange = 2,370 lbs. per inch (Table TB I).

$$\text{Min}^{\text{m}}.\ B = \frac{123{,}000}{2370} = 52 \text{ ins.}$$

$$A_s = \frac{123{,}000}{18{,}000} = 6{\cdot}82 \text{ sq. ins.}$$

9 bars 1 in. (7·06 sq. ins.) in a beam 13 ins. wide would do, or 4 bars 1¼ in. and 2 bars 1⅛ in. in a beam 10 ins. wide. With the former we should have 2·42% of steel with the latter 3·08%.

Using the former with no splay

Quantities per yard run	Concrete $13 \times 22 \times \frac{1}{36^2}$	0·22 cub. yds.
	Steel $18 \times 7{\cdot}06$	128 lbs.
	Centering $\frac{13 + 22 + 22}{36}$	1·59 sq. yds.

The mid-span section will be as Fig. 24.

This example has been run out in detail with full explanations and would be contracted in practice to about half the length. The calculation for finding d for the slab would not be written down (the value of 138 × 12 is 1660 and can be remembered as such). In the secondary beam the flange need not be checked as the designer will see by eye that the provision made is ample. The weights of steel would not be written in detail and if in a hurry the diameters and numbers would be omitted, the areas only being calculated and arrangement of bars, etc., postponed until a final design is made. The use of separate superloads for slab, secondaries and main beams prevents us from writing down the reactions on the plans as the main beam is not designed to carry the full shears calculated for the secondary beams.

As an example of a heavier floor let us take a warehouse floor spanning 16 ft. 0 ins. × 25 ft. 0 ins. for interior panels and carrying a superload of 448 lbs. per sq. ft. with no finishes. Stresses of 750 and 18,000. Depth of construction not to exceed 30 ins. at the outside.

[M on mains say $16 \times 550 \times 25^2 = 5{,}500{,}000$ lbs.-ins.

Say a 5½-in. slab (Fig. 242) spanning 8 ft. 4 ins. (Fig. 244)]

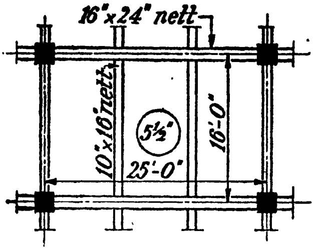


Fig. 244.

Slab span 8 ft. 4 ins. Load = 448 super
69 s.w.

517

$$M = 517 \times 8{\cdot}33^2 \times \frac{12}{12} = 35{,}900 \text{ lbs.-ins.}$$

Min^m. *d* = 4·65 ins. use 5½-in. slab.

$$A_s = \frac{35{,}900}{18{,}000 \times 0{\cdot}857 \times 4{\cdot}75} = 0{\cdot}49 \text{ sq. ins.}$$

½-in. every 4½ ins. = 0·52

0·153 cub. yds.
24 lbs.
1·0 sq. yds.

Secondary span 16 ft. 0 ins. Load = 8·33 × 517 = 4300 lbs.
plus say 150 „

4450 „

$$M = 4450 \times 16^2 \times \frac{12}{12} = 1{,}140{,}000 \text{ lbs.-ins.}$$

Shear = 4450 × 8 = 35,600 lbs.
at 225 lbs. per sq. in = 159 sq. ins.
Using 4 bars 1⅛ ins. (4·0 sq. ins.)

$$\text{lever arm} = \frac{1{,}140{,}000}{18{,}000 \times 4} = 15{\cdot}9 \text{ ins.}$$

Use a beam 10″ × 21½″ overall (10″ × 16″ net)

0·123 cub. yds.
72 lbs.
1·17 sq. yds.

Main Beam span 25 ft. 0 ins. loaded at third points.

Reaction from secondary = 4450 × 16 = 71,500 lbs.
add self-weight say 8·33 × 360 = 3000 „

74,500 „

Positive M say $\dfrac{74{,}500 \times 25}{4{\cdot}5} \times 12$ (compare Table CB IV)

$$= 4{,}980{,}000 \text{ lbs.-ins.}$$

Shear = about 75,000 lbs. requires 333 sq. ins.

Using a beam 29½ ins. deep overall with a lever arm of about 23 ins.

$$\text{Total compression} = \frac{4{,}980{,}000}{23} = 216{,}000 \text{ lbs.}$$

with $d = 26$ ins. about, a 5½-in. flange takes 3100 lbs. per inch.

$$\text{Min}^{\text{m}}.\ B \text{ required} = 70 \text{ ins.}$$

$$A_s = \frac{216{,}000}{18{,}000} = 11{\cdot}9 \text{ sq. ins.}$$

We could use 6 bars 1¼ ins. and 3 bars 1⅜ ins. in a beam 16 ins. wide or 12 bars 1⅛ ins. in a beam 20 ins. wide.

Adopting the former and allowing a steel factor of 19 to cover short additional bars at the support

Concrete 16″ × 24″	0·296 cub. yds.
11·9 × 19 say	230 lbs.
16″ + 24″ + 24″	1·78 sq. yds.

An elevation of the end of the main beam may be very similar to the bottom detail in Fig. 140. The floor is sketched in Fig. 244.

This example illustrates the use of splitting the span into three if possible. Had we used a thinner slab spanning 6 ft. 0 ins. we should have one secondary at mid-span and the load would have been considered as a distributed load. The shear would have been greater on the main beam. The practice of aiming at splitting the main span into three parts is very common in steel but not in reinforced concrete. The reader should work out this example again, using a 4½-in. slab spanning 6 ft. 3 ins.

In considering the load due to three or four secondaries as uniformly distributed, we over-estimate the shear on the main beams and this should preferably be taken from a plan showing the reactions. In the last example worked out where the load is taken as concentrated at two points we have taken one-third of the self-weight of the main, acting at each point. In calculating the total shear we have thus neglected one-sixth of the self-weight of the main beam. This is not a serious matter. As the self-weights are estimated before the member is designed they will seldom be exactly right. This may be corrected when doing the final design but is quite negligible in all preliminary designs.

The reader will see that little difficulty arises in calculating the members for a preliminary design once the spacings are fixed. The one great point is to make certain that the most economical spacing has been found. For this purpose it is as well to try two or three alternatives, using different thicknesses of slab. At the present time when relative prices change rapidly this is always desirable. Some firms run out the quantities as so much per square yard of floor including beams. This should always be carried out by taking off the whole floor and all beams and then dividing by the total area. It should not be done by trying to average out the beams according to spacings, allowing a bit extra for the bressummers. Such average quantities are always useful as experts can tell at a glance whether they are approximately correct or not.

Columns.—The columns are designed by drawing a vertical double line to represent each, or each group of similar columns. The loads at each floor are written on by drawing horizontal lines across at intervals to represent the floors. The total load on each length is then written opposite the centre of that length half-way between the lines representing the floors. The sections may be calculated or taken from tables (Chapter VI). An effort should be made to give all columns in the same row the same overall size, even if the loads vary slightly, by using different percentages of steel. This improves the appearance and makes the column boxes interchangeable.

Some designers leave a left-hand margin in the calculations in which they put all main sizes. Some others underline main sizes in red pencil, etc.

CHAPTER XXIX

CONTINUOUS FOOTINGS

When the average load per square foot of any structure approaches the safe bearing power of the site then the footings may be combined into one continuous footing sometimes called a raft. The case of a continuous footing beam for a single row of columns has been discussed in Chapter XII.

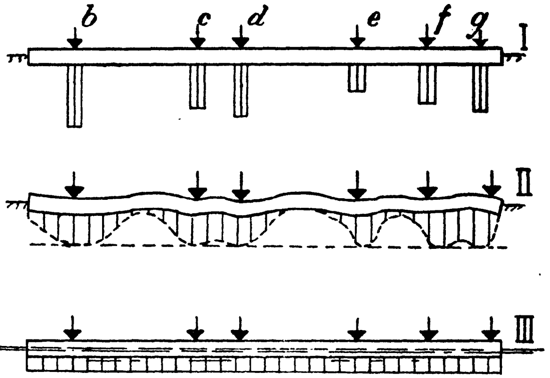


Fig. 245.

Loading.—Some authorities have stated that a raft foundation is an inverted floor. This is entirely wrong in principle (see the end of Chapter XII). A raft footing looks like an inverted floor when constructed and the bending moments under full load resemble those on a floor, but the underlying principles are different.

A section through a continuous footing is shown in Fig. 245. In case I we have a series of loads *b*, *c*, *d*, etc., acting at points on the upper surface of the raft, counterbalanced by small loaded areas underneath. In this case the loading underneath the raft occurs in small patches of heavy pressure. Case III shows a raft actually floating in some heavy liquid, it being assumed that the centre of load is somewhere near the middle and that the whole raft will not capsize. In case III the pressure on the under side is uniform provided the raft stays level, and proportional to the depth immersed. Case I illustrates what happens when a raft is placed on solid rock. The load on each column passes directly through and causes no appreciable stresses on the raft outside the areas of the columns. Case III illustrates the equilibrium of a floating caisson. Case II is intermediate and illustrates what actually happens in a continuous footing on a yielding foundation. The soil is unable to bear the heavy pressures produced in case I and begins to yield. The raft commences to settle and deflect, the settlement causing compression on more extended areas until the raft is in equilibrium.

It is well known that wet clay will support a certain definite pressure. If loaded beyond this pressure the clay passes into a state of flux and yields indefinitely. This pressure, which is known as the flow pressure*, is what limits the bearing power of a raft. When a point load is applied to the upper surface of a raft foundation it tends to pass through and create a small patch of high pressure as in case I. If this intensity is beyond the flow pressure of the soil then movement commences. We may look upon case I as an initial condition of case II before the material has had time to flow. The amount of flow is small (a factor of safety is of course allowed in estimating the safe bearing power of the site) but the fact that large buildings on rafts have been known to settle at the rate of 12 ins. in 10 years or so, seems to place it beyond doubt that flow does take place. (In the case of Winchester Cathedral the settlement extended over centuries.) The problem to be solved is to provide sufficient area in the neighbourhood of each column base that the load on the column may spread over the ground at an intensity not exceeding the safe pressure, to calculate the stresses, etc., produced by such spread of load and to provide a raft capable of carrying them. In fact, we may look upon a raft footing as the limiting case of a number of isolated column footings, the size of the footings being increased until each footing touches its neighbours. If the columns are all evenly spaced and evenly loaded at full load then the pressure under the raft approaches the full allowable pressure and the raft does assume the conditions of a uniformly loaded floor. With other types of loading it is possible to get very

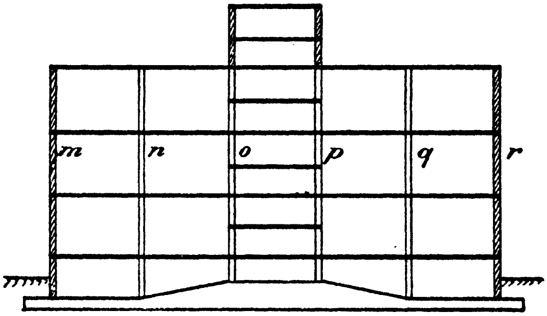


Fig. 246.

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* Min. Proc. Inst. C. E. Vol. CCXIII, p. 144 et seq.

different results. We may assume that flow takes place when the intensity reaches the allowable pressure. This is not strictly true but is done to allow a factor of safety between the actual flow pressure and the allowable bearing pressure.

If, therefore, one column or group of columns is more heavily loaded than the surrounding columns, provision must be made to spread the load beyond the neighbouring panels of raft.

The case of a building having six rows of columns is shown in Fig. 246, which is a section through the building. Columns *o* and *p* are more heavily loaded than columns *n* and *q* owing to the addition of a tower, provision of mezzanine floors, heavier superloads, etc.

If the area of raft under each column is not sufficient to carry this heavy load then some of the load on *o* and *p* must be spread beyond the columns *n* and *q*, and a construction similar to that indicated in the figure will have to be provided. (The total load on the whole structure must, of course, be less than the safe bearing pressure over the whole site, or else the whole building will be in danger of sinking bodily, however strong and stiff the raft is.)

Choice of Type of Raft.—There are three types of raft :—

(1) Plain slab.
(2) Flat slab.
(3) Slab-and-girder.

The plain-slab raft consists of a thick slab of concrete of uniform thickness, reinforced with bars both ways at top and bottom. The column bases may project above the general level of the top of the raft or may be flush, with extra bars in the raft immediately under the columns.

It forms a very useful type of raft where the loading is very irregular, and its thickness and the amount of reinforcement are generally put in by experience. Sometimes a rough idea of the bending moments may be found by taking the largest regular panels and taking $\frac{wl^2}{16}$ in both directions. For a single storey building a thickness of 6 ins. with a bit extra under the columns is quite usual with ½-in. bars at 6-in. centres both ways top and bottom. For a somewhat heavier service, say a heavy machine shop or boiler house, 9 ins. to 12 ins. may be used. A seven-storey office building or four-storey warehouse may have a raft 3 ft. 0 ins. thick. Flat-slab rafts are suitable for cases having evenly spaced columns, which are all equally loaded, on sites where no subsidence whatever is to be expected. A flat-slab construction has no strength when looked upon as a beam spanning from one side of the building to the other. If one bay of the building were much more heavily loaded than the rest or if one patch of the site proved softer than expected, the whole basis on which flat-slab design is founded, strong and unyielding support at the column heads, would collapse. If the building is irregular in plan or unevenly loaded when fully loaded (uneven loading under partial loads is not dangerous) or if buildings on neighbouring sites are known to have sunk after completion, then flat-slab construction should be avoided. The small bending strengths provided in Chapter XXIV would be totally inadequate if one column sank below its neighbours.

Slab-and-girder construction offers the most reliable construction in cases of buildings of the warehouse type. Sometimes the beams are placed underneath the slab. On wet sites this is not always possible, as trenches must be dug and kept true to size and free from water. Another objection to beams placed underneath is the tendency of the slab to tear away from the beams. Allowing the beam ribs to project upwards, the raft having a flush under surface, offers the easiest and soundest construction but takes away from the available height of the building, unless the whole raft is put in at a lower level, which necessitates more excavation. In designing rafts the shear presents more difficulty than the bending moments and a mesh panel design, having main beams running in both directions, offers a great advantage in this respect. Another strong point in favour of main beams in both directions is the strength and stiffness of the raft as a whole and its ability to carry over unsuspected soft patches. This type of raft will be adopted as the standard type in the discussion which follows, although the general arguments will apply to all rafts.

Choice of Spacing.—The spacing of the raft members is governed by the spacing of the columns in the structure above. If the engineer is allowed to choose the spacings in the building a square panel will be aimed at. In this respect it may be mentioned that a monolithic building will tend to assist the raft in distributing uneven loads, but this effect should not be counted on unless the columns, etc., are designed for the extra bending stresses. These are so complicated that all the bendings, etc., are invariably assumed to be carried by the raft alone. If subsidence is expected the floors will be better in slab-and-girder construction than in flat-slab, as the former appears to be less affected by sinkage of columns.

If long narrow panels are unavoidable then secondary beams may be used, care being taken to make the slab thick enough to act as compression flange to the main beams.

Loading and Stresses.—From the nature of the problem it is impossible to load any one column fully without at the same time putting a large amount of load on the surrounding columns. The reaction of the earth on the raft is equal to the weight to be carried, and a panel of the raft will not receive its maximum upward thrust from the earth unless all the four columns which stand at the corners of the panel are fully loaded. This means that all the surrounding panels have a large amount of live load reaction as well. The conditions of loading which give the maximum moments on continuous beams are not possible on rafts (see Chapter X) and yet the conditions are not quite equivalent to all dead load. Considering that the load on one column is the sum of a number of floor loads it would be necessary for the superload to be distributed in exactly the same way on each floor to produce the maximum unevenness of loading on the ground. This is extremely unlikely. It must be clearly understood that the author is now dealing with possible variations from the maximum loading due to uneven distribution of superload on the floors. Inequality of loading on the site due to one part of the building having more storeys, or a heavier superload or such intrinsic causes must always be provided for. The point at issue becomes finally this. If the designer will take the whole structure and place on it the maximum superload at all points at the same time, then this arrangement of loading will cause the maximum possible stresses in all members of the raft. This is quite true enough for purposes of practical design.

Process of Design.—The column loads from the structure above must first be run out, the proportion of live load and dead load being noted. The self-weight of the raft itself and anything carried directly on the raft and not included in the column loads must be estimated next. (A large mesh panel raft with spans 23 ft. 0 ins. × 23 ft. 0 ins. with a safe load of $\frac{3}{4}$-ton per sq. ft. weighs about 400 lbs. per sq. ft. of area. A similar raft with spans 18 ft. 0 ins. × 18 ft. 0 ins. weighs 340 lbs. per sq. ft. of area.

Limitations of depth and change of design will cause large fluctuations but these two figures may help the designer to form a preliminary estimate.)

A plan of the raft is then drawn showing the proposed system of beams, and the available area in the neighbourhood of each column is worked out. The interior columns are treated first, as the areas allotted to the outside columns depend on the extent to which the raft projects beyond the external walls (if any). The safe bearing capacity of the area under each column is worked out and compared with the total load on that area. The extent to which the cantilever toe of the raft should project beyond the external walls will be decided by the loads on the external columns. Having decided all this, a table showing the safe bearing of each column area, the actual load on the area and the excess of one over the other may be drawn up. The various columns in the table should be added up to show the total load on the whole structure and the total safe bearing capacity of the site. It is as well to obtain exact values of the column loads before starting to design the raft. The reader will find that approximate values of column loads are always on the low side owing to the omission of an allowance for internal walls, etc. In working out the areas needed by the external columns it should be remembered that these carry the external walls and have a larger percentage of dead load. The superloads may or may not be realized but the dead loads are, of course, certain to act. If the safe bearing capacity of each column area exceeds the load on the column then the case is a simple one. The load on each column is transferred to that part of the site in its immediate neighbourhood and to no other part of the raft. If the column load in any case exceeds the safe bearing of the column area then the excessive intensity of pressure will cause the ground to yield and the pressure will spread laterally causing the beams, instead of acting merely as local distributors of pressure, to span much longer distances and transfer the excess load to areas which are more lightly loaded. As an alternative a system of friction piles may be introduced, grouped more closely under the more heavily loaded portions. Friction piles will obtain some grip even in the worst building site and tend to prevent or retard settlement by making the flow of earth take a deeper course. In a mesh panel raft the piles may be strung out in single or double lines under the beams.

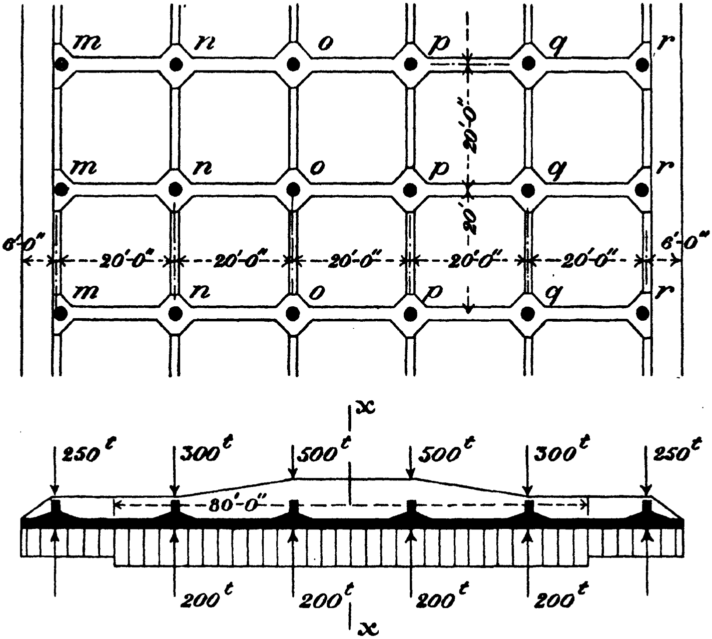


Fig. 247.

In order to illustrate the point we will take the case of a warehouse building on a site which will safely carry one ton per square foot and assume a mesh panel raft, the columns being spaced at 20 ft. 0 in. centres in both directions. We shall thus have an area 20 ft. 0 ins. × 20 ft. 0 ins. under each internal column. This area will support 400 tons. If the load on the internal columns (including the self-weight of the raft, etc.) does not exceed 400 tons, then each area is quite capable of supporting its own column and the beams will have local bending moments only. If, however, we have the case shown in Fig. 247 the beams may also carry a general bending moment.

Fig. 247 shows a part plan corresponding to the section in Fig. 246. Owing to the extra floors the loads on columns *o* and *p* in the centre rows are 500 tons per column. The load on columns *n* and *q* are 300 tons per column while the loads on the outside columns are 250 tons per column. The columns are spaced at 20 ft. 0 ins. × 20 ft. 0 in. centres, giving an area of 400 sq. ft. for each internal column. The raft has a cantilever toe projecting 6 ft. 0 ins. beyond the external columns which have an area of raft of 320 sq. ft. to each. The ground will carry one ton per square foot safely. Columns *o* and *p* will thus have 100 tons more load than can be carried by the area of raft allotted to them while columns *n* and *q* have 100 tons less. This might be remedied by putting friction piles under beams *o—o—o* and *p—p—p* etc., otherwise beams *mnopqr* must be strong enough to spread the extra load beyond columns *n* and *q*. If neither of these expedients is adopted then the intensity of pressure under columns *o* and *p* becomes excessive, the ground reaches its " flow pressure " and keeps on sinking until the raft breaks its back. In the case in point if we spread the load as shown in the section in Fig. 247, the intensity of load under the middle eighty feet of the building being exactly one ton per square foot, we shall be just safe having a load of 1600 tons on an area of 1600 sq. ft. The intensity of pressure under the outer columns will be less and we may assume that the reaction of the ground under these columns exactly balances the column load. Under this loading beams *o — o — o* and *p — p—p* carry an upward load of 10 tons per ft. run from the ground (see Chapter XVII) and thus bring an upward shear force of 200 tons under each column *o* and each column *p*. Beams *n — n — n* and *q — q — q* are closely similar. (This case is only cited to illustrate the argument. Practical cases are always complicated by questions of the self-weight of the raft, uneven spacing of columns, etc. As these complications do not concern the underlying principles they are not taken into account in this discussion.) Beams *n — o*, *o — p* and *p — q* carry also a load of 10 tons per ft. run. The whole loading is shown in the section in Fig. 247. Taking moments for section *x — x* and assuming that the load on the exterior column is exactly balanced by the ground reaction (this is not strictly correct).

$$\begin{aligned} M &= 200 \times 10 + 200 \times 30 + 400 \times 20 - 300 \times 30 - 500 \times 10 \\ &= 2000 + 6000 + 8000 - (9000 + 5000) \\ &= 2000 \text{ tons-feet.} \\ &= 53{,}800{,}000 \text{ lbs.-ins.} \end{aligned}$$

The beam at section *x — x* will be designed as a rectangular beam to resist this moment by the principles in Chapter XIII. If the section is 24 ins. wide by 96 ins. overall with $d = 90$ ins.

$$R = \frac{53{,}800{,}000}{24 \times 90 \times 90} = 275.$$

This will require about 2% of steel top and bottom. Coming next to the question of local bending moments. The slab is fully loaded when carrying the allowable pressure of 1 ton per sq. ft. (The self-weight of the slab and beams will have to be subtracted from the upward pressure when calculating the moments.)

If the internal panels are nearly square and W is the effective load on each, the slab may be splayed out at the beam as shown in Figs. 247 and 248 (or stepped up) and designed for a moment of $\frac{W}{72}$ in the centre of the panel and $\frac{W}{36}$ at the beams.

The cantilever toe, if consisting of a slab only, should be symmetrical about the outside beam, the bars being taken well into the first square panel. The stiff toe will fix the edges of the outside panels which may be calculated as similar to the internal ones.

It will be seen that the task of spreading over-loads falls on the beams and it is as well for the stiffness of the raft as a whole, to aim at some uniform beam construction throughout. The size of the beams is largely settled by questions of shear which may be helped out by splaying the beams in plan as indicated in Fig. 248. In Fig. 247 the beams $o-o-o$ and $p-p-p$ carry a uniform load of 10 tons per ft. run (less their self-weight). On a foundation where no settlement will take place these beams may be designed for moments of $\frac{w\,l^2}{24}$ between columns and $\frac{w\,l^2}{12}$ under the columns. Where settlement of any kind is expected $\frac{w\,l^2}{12}$ should be used at both places, aiming to use as long bars as possible and keeping a fair area of steel at all sections with a stout row of vertical stirrups evenly spaced to help take shear in either direction. In the case of accidental local settlement due to an unnoticed soft patch any convexity downwards would be assisted by tensile stresses in the bars of the slab and compression in the upstanding ribs of the beams. Convexity upwards would be resisted by tensile stresses in the tops of the ribs only, which should therefore never be devoid of steel at any point, whatever the calculated local moments may be. In the case illustrated in Fig. 247 beams $o - o - o$ and $p - p - p$ will have a moment of

$$\frac{w\,l^2}{12} = 10 \times 2240 \times 20^2 \times \frac{12}{12} = 8{,}930{,}000 \text{ lbs.-ins.}$$

and a shear of 10 × 2240 × 10 lbs. at the column.

Allowing for the splay cutting off the last 3 ft. 0 ins. of beam the shear on the minimum cross section will be

$$10 \times 2240 \times 7 \text{ lbs.} = 156{,}000 \text{ lbs.}$$

At 300 lbs. per sq. in. (the beam is deep for its span) this would require 520 sq. ins.

Using a beam 14 ins. wide with three rows of bars this would require an overall depth of about 48 ins. The bending moment would call for 13 sq. ins. of steel at mid-span using $\frac{w\,l^2}{12}$.

Considering the service required of a raft and the disastrous consequences of even a local failure a little liberality in design seems to be called for, particularly where the raft carries an expensive office-building with a masonry front which shows up at once if relative settlement occurs. In running out the number of piles to be used in a mixed scheme the designer should allow for rational superloads to keep all settlement as even as possible. Portions of the raft carrying very light loads such as the centres of large open areas inside a building, should be loaded with ballast to an equivalent weight. It is good practice to design small bays of floor for 75 to 100 lbs. to the square foot for an office building but such loads are impossible over large areas. In a large building 200 ft. 0 ins. × 300 ft. 0 ins. in plan, having five floors, a distributed superload of 100 lbs. per sq. ft. is 30,000,000 lbs. or 13,500 tons, equal to the weight of about 200,000 people. In such a case 10 lbs. to 20 lbs. per sq. ft. over the whole area would be ample.

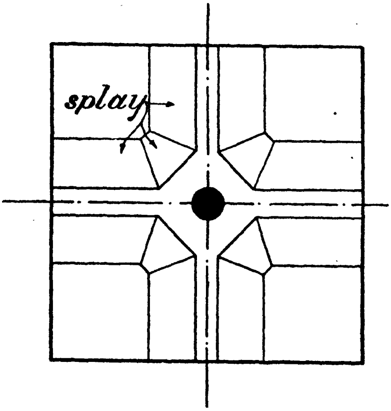


Fig. 248.

Rafts Conclusion.—Most practical cases show more or less difficult complications and a carefully drawn plan with all loads and areas marked on will assist greatly in the design. The author likes to look on a raft as a series of isolated footings, as shown in Fig. 248, and the extent of the footing under each column should be drawn on the plan, together with the safe bearing pressure on the soil for each footing. If the load on the column exceeds the safe bearing, then the excess load should be transferred to another plan which shows the excess loads only, cases where the column load falls below the safe load being reckoned minus. Turning our attention to this latter plan we have a series of plus and minus excess loads. The plus loads must be transferred to minus areas or neutralized by piles and any minus loads, left over after the operation, must be neutralized by adding ballast provided they are of sufficient value to cause anxiety about unequal settlement. The bending moments on the raft are due to two causes therefore :—

(a) A uniformly distributed load over the whole site causing local bending only.

(b) An additional series of moments, etc., due to transference of load from one area to another or due to the additional upward reaction provided by the necessary piles.

These moments may be calculated separately and then added.

The author would again point out that a raft does not float in the sense that a body floats in a pure liquid. Transference of load from one part of the site to another only takes place when the intensity of local loading exceeds the " flow pressure".

The quantities for a raft may be roughed out by taking a typical bay if the arrangement of bars in the beams is unusual, and steel factors cannot be used. Special allowance must be made for the cantilever toe which contains a surprisingly large percentage of the total steel.

CHAPTER XXX

WATER-TOWERS. RESERVOIRS

Small Towers.—Of about 10,000 gallons capacity are usually square boxes on four columns. The sides and bottom and roof (if any) are mesh panels. If P is the total water pressure on one side it may be designed for a moment of $\frac{P}{30}$. Do not forget to add steel to take the direct bursting tension.

The floor panel may be supported by using the walls as vertical girders but a more watertight job is likely if beams are provided under the floor. The wall panels are supported by one another, by the bottom panel and by the roof (if any). If there is no roof a horizontal beam is required round the top of the walls. For a mesh panel carrying a triangular load, the shears round the edges may be taken as shown in Fig. 249 (this is a very crude approximation but amply good enough in practice).

Fig. 250 shows two half-sections with the variations described above.

Medium Towers.—Towers of 30,000 gallons to 100,000 gallons generally have circular tanks and are supported on four to eight columns. The tank roof carries little load and a couple of beams each way will suffice. The smaller tanks may be roofed over with a plain mesh panel slab designed for a moment of $\frac{W}{18 \cdot 8}$ where W is the total roof load. The roof slab may be built as a dome but this is neither useful nor economical.

The ratio of diameter of tank to depth of water may vary from one to two and the most economical arrangement is generally somewhere between these figures. When commencing a design we may begin by trying a tank where the diameter is one and a half times the depth. The design of circular tanks is in Chapter XX.

There are several ways of constructing the bottom, some of which are shown in Figs. 251 to 253.

Fig. 251 is an early design of French origin. The bottom is divided into sectors by a number of radial beams, the column being placed under the centre of gravity of each sector (when the tank is full). The floor slab is cut up into irregular panels, the exact position and amount of the maximum moments being rather problematical. The easiest solution is to take the largest panel and estimate the maximum moment.

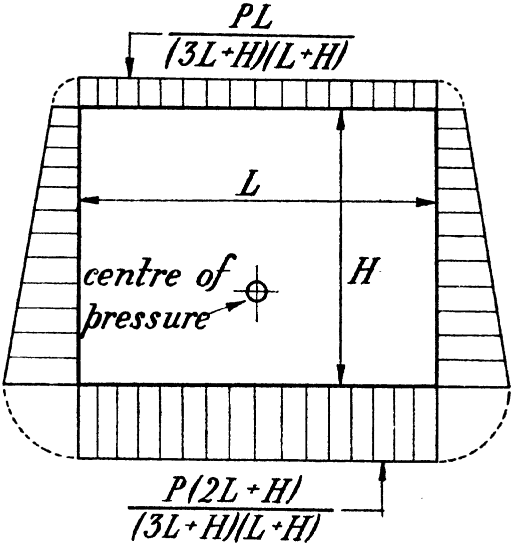


Fig. 249.

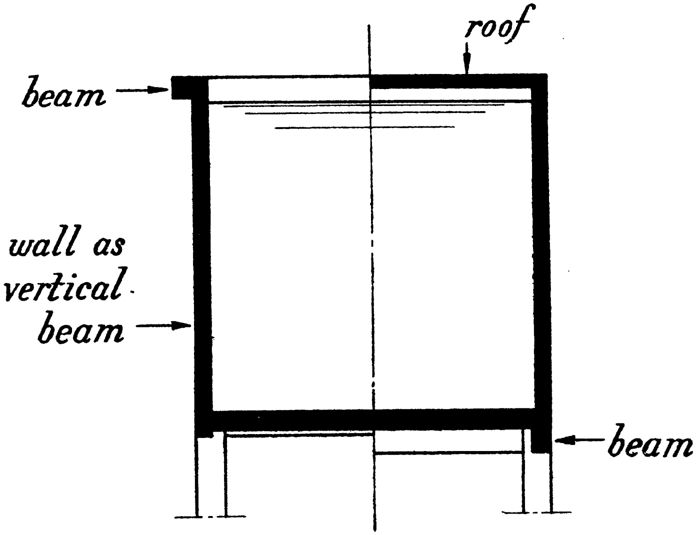


Fig. 250.

Design the slab to carry this and supply steel top and bottom both ways throughout, using straight bars only. As has already been shown a system such as this will resist moment of either sign in any direction. In apportioning the load to the beams the best way is to draw a plan of the bottom to scale and draw in the areas allotted to each beam from first principles. It will be seen that the circular wall of the tank is called on to act as a beam and support the outer edge of the outermost panels of floor. The circular wall is in turn supported on the outer ends of the cantilever main beams and the shear produced at this point requires adequate provision. The outside of the tank wall is usually " ornamented " by two cornices, one near the top and one near the bottom. The bottom one should be placed at the same level as the floor so that the floor bars may be run through into it.

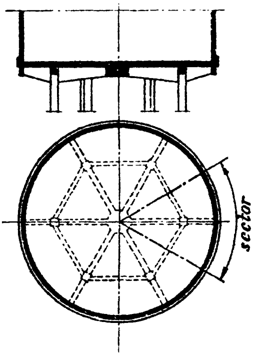


Fig. 251.

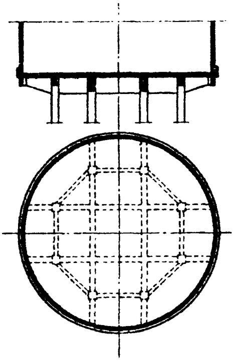


Fig. 252.

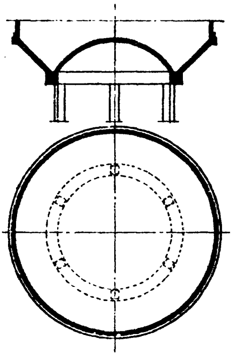
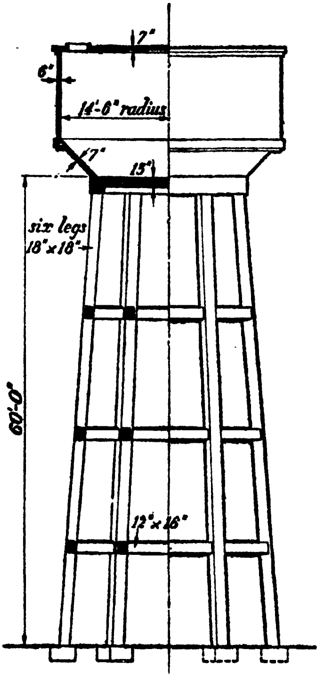


Fig. 253.

Fig. 252 is a modification of the arrangement in Fig. 251. The tank bottoms in Figs. 253 and 254 are brought in in the form of a cone, the centre part in Fig. 253 being a dome while the centre of Fig. 254 is a mesh panel. Taking everything into account Fig. 254 is generally cheaper than Fig. 253. The conical portions must be designed as described in Chapter XXV.

The columns may be either vertical or battered. For a high tank battered columns give a larger base, but for a low tank, where a large base is not required, vertical columns are cheaper and give shorter braces. Increasing the number of columns makes very little saving in the tank floor and the smallest practicable number of columns should be used. The relative merits of square, hexagonal and octagonal columns are discussed in Chapter VIII. The braces should be spaced at 15 ft. 0 ins. to 20 ft. 0 ins. vertically. A wind pressure of 20 lbs. per sq. ft. on the circular tank and 30 lbs. per sq. ft. on the columns and braces may be taken. Judging by some of the earlier towers still standing this allowance appears ample. The column footings on good ground may be separate footing slabs which must be large enough to keep the line of resultant pressure, after



60,000 GALLS. TOWER

Fig. 254.

allowing for wind bending, always inside the base. On poor ground a continuous footing beam makes a sound design.

In splitting up the wind bending moments between the various braces that meet at a column (see Chapter XIII) the angles they make with the assumed direction of the wind must be taken into account.

For four, six or eight columns the bracing shown in plan in Fig. 255 may be used.

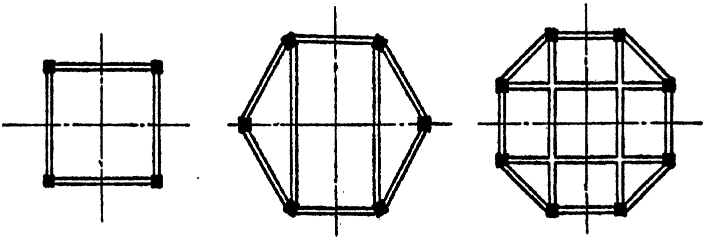


Fig. 255.

If cost is no object, a reinforced concrete wall may be substituted for the ring of columns and braces. In some American designs a dished suspended floor has been adopted. (These are treated in Chapter XXV.)

Large Towers.—An arrangement for the floor of a tower of 200,000 to 500,000 gallons capacity is shown in Fig. 256 supported on two rings of columns, twelve in the outer ring, six in the inner ring and one in the centre.

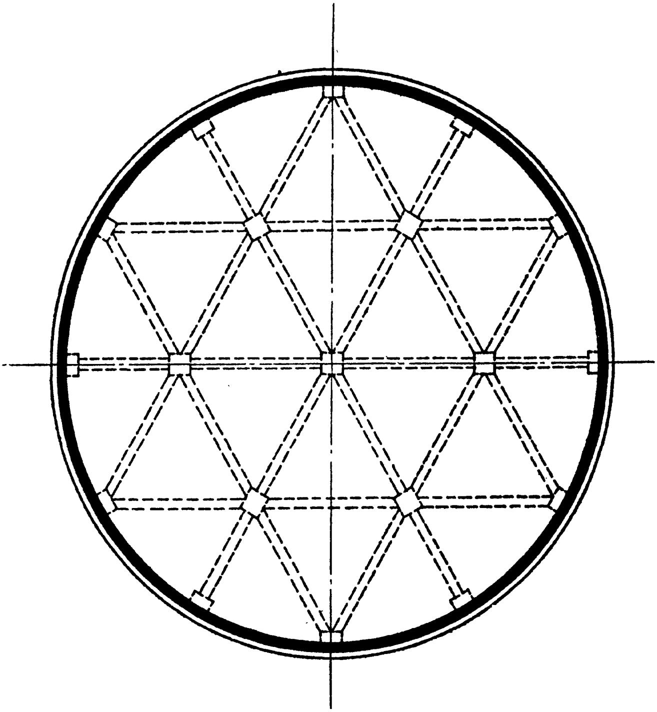


Fig. 256.

Very Large Towers.—Elevated tanks of 1,000,000 gallons upwards are usually rectangular in plan with rows of columns in both directions. The walls will be similar to those of small reservoirs, the tank floor being mesh panel or flat-slab.

Reservoirs.—The question of design is tied up with the ground levels, water levels, and nature of the foundation.

On some sites it is both possible and economical to increase the capacity by excavating and taking the reservoir floor well down below ground level. On others the loss of head, cost of excavation, or natural ground water level may necessitate keeping the tank floor as high up as possible. On sound and dry foundations the floor may be 5 ins. or 6 ins. of reinforced concrete (laid on 2 ins. of mass concrete) containing 10 to 15 lbs. of steel per sq. yd. On a poor foundation or on a steep slope the floor may be supported on piles or on columns or on mass concrete blocks or sleeper walls. On a wet foundation, say a permeable bed of gravel, the tank floor may be subjected to upward pressure from the ground water when the reservoir is empty. Unless the floor is sufficiently thick and heavy to counterbalance this pressure it will have to act as a reversed floor spanning from column to column, the columns being held down by the weight of the roof and superimposed load. The alternative solution of putting relief valves in the floor is scarcely practicable in drinking water reservoirs and there is always the chance that such valves may give trouble after some years of service. If no roof is provided the walls will act as vertical retaining walls (see Chapter XXXIII). If a roof is provided, the walls may span vertically, being tied in by the floor and roof. For a tank 10 ft. 0 ins. to 15 ft. 0 ins. deep a simple slab spanning vertically will suffice while for greater depths vertical counterforts with a slab spanning horizontally between them will be cheaper (or vertical counterforts and mesh-panel slab walls). In a narrow tank ties running from wall to wall will tie the walls together if no roof is provided.

Walls of reservoirs should be designed to take full water pressure outwards or full earth pressure inwards and the reservoir should be filled and tested before any earth backing is thrown up against the walls. For a plain slab spanning a distance L vertically, fixed at the bottom and resting against an immovable support at the top, carrying a triangular load P the reaction at the top is $0{\cdot}2\ P$ the reversed moment at the bottom is $-\ 0{\cdot}133\ PL$ and the positive moment in the span is $+\ 0{\cdot}0595\ PL$. In practice the stretching of the roof due to applied tension or opening of construction joints or the shrinkage of the roof due to its being dry while the walls are wet, or slight movement of the bottom of the wall will upset the theoretical moments and these should be increased to cover the uncertainties.

With vertical counterforts and a slab spanning horizontally the moment on the slab varies as the depth of water and the slab may splay out from 5 ins. or 6 ins. at the top to a maximum at the bottom. Although the slab is supposed to span horizontally from counterfort to counterfort the bottom two or three feet will actually cantilever off the bottom slab in a manner similar to the bottom part of the wall in a circular tank in the problem of restraint in Chapter XX. The counterforts themselves are usually simply supported at top and bottom. Although the loading from the water pressure on the slab is fairly definite it is usual for the sake of watertightness to design the slab for moments of $+\frac{pl^2}{12}$ and $-\frac{pl^2}{12}$ and not $+\frac{pl^2}{24}$ and and $-\frac{pl^2}{12}$ as might be thought.

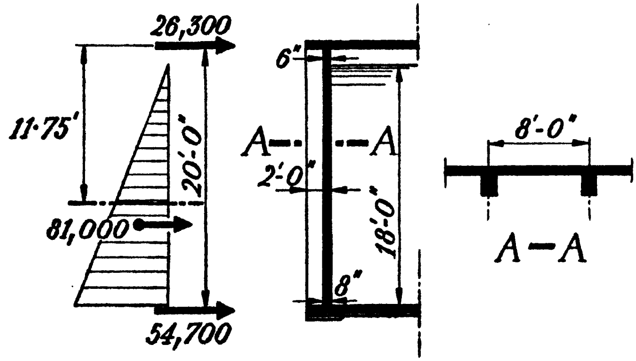


Fig. 257.

Example.—Fig. 257 shows a section through a reservoir wall retaining 18 ft. 0 ins. of water. The slab spans horizontally from counterfort to counterfort, these being spaced at 8 ft. 0 ins. centres. The counterforts span vertically a distance of 20 ft. 0 ins. Working stresses are 750 and 16,000 lbs. per sq. in. with $m = 15$. (With these values $n = 0{\cdot}414$, $j = 0{\cdot}862$ and $R = 134$.) Allowing for the fact that the extreme bottom part of the wall slab is anchored to and cantilevers from the floor slab we may assume that the worst moment in a horizontal direction occurs at a depth of 16 ft. 0 ins.

$$\text{pressure} = 62{\cdot}5 \times 16 = 1000 \text{ lbs. per sq. ft.}$$

$$M = 1000 \times 8^2 \times \frac{12}{12} = 64{,}000 \text{ lbs.-ins.}$$

$$\text{Min}^{\text{m}}.\ d = \sqrt{\frac{64{,}000}{134 \times 12}} = 6{\cdot}3 \text{ ins.}$$

If we make the wall slab 6 ins. thick at the roof, tapering uniformly to 8 ins. at the floor it will be about 7·8 ins. thick, 16 ft. 0 ins. below water level. Allowing 1 in. cover, this will give a value of about 6·5 ins. for d at 16 ft. 0 ins. depth.

$$A_s \text{ at } 16 \text{ ft. } 0 \text{ ins.} = \frac{64{,}000}{16{,}000 \times 0{\cdot}862 \times 6{\cdot}5} = 0{\cdot}712 \text{ sq. ins.}$$

Say ⅝-in. bars at 5-in. centres.

This steel will be reduced as we go farther up the wall and the pressure diminishes. If, as is usual, earth is banked up against the outside of the wall, steel must be provided on the inside face to resist the earth pressure.

The counterfort carries pressure from 8 ft. run of wall or

$$8 \times \tfrac{1}{2} \times 62{\cdot}5 \times 18^2 = 81{,}000 \text{ lbs.}$$

The centre of pressure is about 6 ft. 6 ins. above the lower point of support the reactions top and bottom being 26,300 and 54,700 lbs. The maximum moment occurs 10·25 ft. below water level or 11·75 ft. below the top support.

Max^m. moment = 26,300 lbs. × 8·32 ft.
= 2,630,000 lbs.-ins.

Using a counterfort 24 ins. deep overall, $j\,d$ = about 19 ins.

$$A_s = \frac{2{,}630{,}000}{16{,}000 \times 19''} = 8{\cdot}65 \text{ sq. ins.}$$

4 bars 1¼ ins.	= 4·91 sq. ins.
4 bars 1⅛ ins.	= 3·98 ,, ,,
	8·89 ,, ,,

Maximum shear at bottom = 54,700 lbs.

Taking all this on the steel

4 bars 1⅛-ins. at 30°	$= 3{\cdot}98 \times \frac{1}{2} \times 16{,}000$	= 31,800 lbs.
Double ⅜-in. links at 5 ins.	$= 4 \times 0{\cdot}11 \times \frac{19}{5} \times 16{,}000$	= 26,800 lbs.
		58,600 lbs.

Steel must be provided on the inside of the counterfort to resist the earth pressure (if any) when the reservoir is empty. Details are shown in Fig. 258.

The direct tension in the bottom of the tank due to the reaction from the counterfort is soon neutralized by frictional resistance between the floor and the foundation. The direct tensions in the roof go further. Full provision must be made to take them in the outside panels but they will be taken up by the general reinforcement

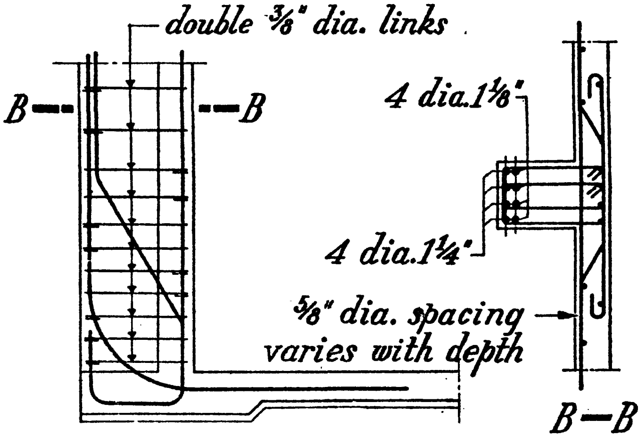


Fig. 258.

in the centre part of the roof which is unlikely ever to take its full calculated load.

The roofs of reservoirs are usually of flat-slab construction or sometimes mesh panel. Deep narrow beams hanging down close to the water surface usually start to deteriorate after five or six years service.

An unusual but very successful form of construction where the walls are horizontal arches is shown in Fig. 259.

For a discussion on reinforced concrete suspension flumes the reader is referred to the *Engineering News* of April 8th, 1915, by H. M. Gibb.

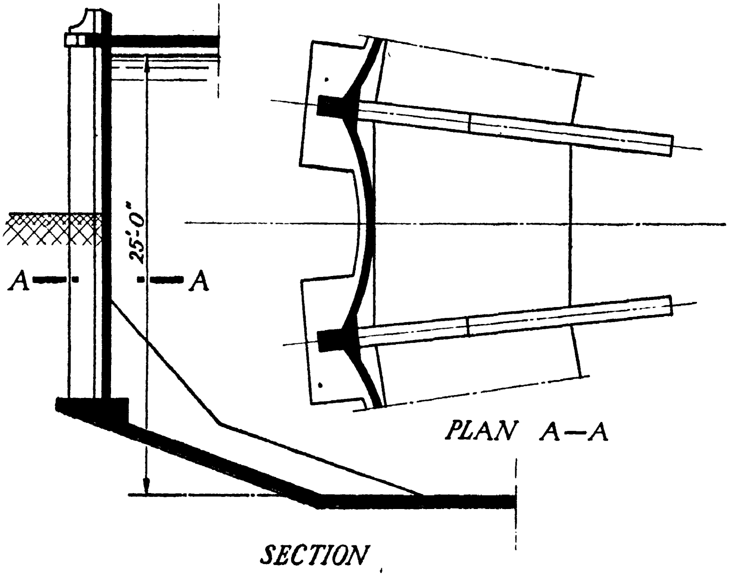


Fig. 259.

CHAPTER XXXI

BUNKERS AND SILOS

Bunkers.—General Arrangement.—The choice of type is not often left to the designer, but an effort should be made to introduce a simple type of hopper bottom (see Chapter XXII). If the bunkers are very wide, the bottom may be broken up by a system of beams and rows of intermediate columns.

The general design of bunker walls is controlled by the cost of the shuttering. A high vertical panel of wall slab can be made with one set of shutters at a cheap rate. A panel which is broken up into small heights by horizontal beams is more expensive both in shuttering and steel fixing. Usually then we find the slab spanning horizontally between vertical counterforts (or cross-walls). In coke bunkers especially, where the pressure is small, large mesh panels give an attractive solution, but the closely spaced long vertical bars need supporting by scaffolding and make it difficult to get from one side of the wall to the other. These difficulties are, of course, not insuperable but an item should be allowed in the estimate to cover them.

Pressures and Spacings.—Dry coke is usually taken as weighing 28 lbs. per cub. ft., with a lateral pressure of 5 to 6 lbs. per cub. ft. Coal at 56 lbs. per cu. ft. with a lateral pressure of 10 to 12 lbs. per cub. ft. (Other materials are covered by Chapter XXXIII.)

Small bunkers are usually designed without cross-ties but the walls of large bunkers may be considerably cheapened by introducing tie-beams. The more ties used the cheaper the walls. On the other hand too many ties may interfere with the flow of coal. The minimum practical thickness for a vertical slab is 4½ ins. (5 ins. for mesh-panels). The designer may begin by seeing how far a 4½-in. slab will span or how thick a mesh panel wall would be. Suppose we have a bunker 25 ft. 0 ins. long containing coal with a lateral pressure of 12 lbs. per cub. ft. and the maximum head of coal is 20 ft. 0 ins. The maximum lateral pressure is then 240 lbs. per sq. ft. Using a 4½-in. wall with ¾-in. cover and working to 750 and 18,000 the slab will resist a moment of

$$138 \times 12 \times 3{\cdot}5^2 = 20{,}300 \text{ lbs.-ins.}$$

If l is the maximum possible span

$$240 \times l^2 \times \frac{12}{12} = 20{,}300$$

$$\text{or } l = 9{\cdot}2 \text{ ft.}$$

We may divide the 25 ft. 0 ins. into three spaces of 8 ft. 4 ins. If we have a coke bunker whose side is 16 ft. 0 ins. wide and 16 ft. 0 ins. deep with a level surface the total pressure at 5 lbs. per cub. ft. is $5\times16\times\frac{16^2}{2}=10{,}300$ lbs.

$$M=\frac{P}{30}=\frac{10{,}300}{30}\times12=4120 \text{ lbs.-ins.}$$

A 5-in. mesh-panel ($d = 3{\cdot}75$ in.) will easily take this.

Vertical Loading.—In most bunkers of small or moderate size (say less than 1000 tons) it is possible to utilize the side walls to act as vertical girders to suspend the hopper bottoms. In large bunkers the shears are so heavy and the large side walls are cut into so many pieces by construction joints and (with modern cements) may have such heavy shrinkage stresses that it is sounder design to provide beams to carry the vertical loads and not to use the side walls for this purpose.

Design of Wall Slabs.—If the wall slab spans horizontally it is usually made of uniform thickness all the way up. In this respect it differs from reservoir walls (see Fig. 257) or from the front slabs of large counterfort retaining walls (see Fig. 272) which generally taper from bottom to top. The main reason for the difference is the much smaller lateral pressure as bunkers are usually built to hold coal or coke. For bunkers to hold heavy ore, construction similar to that of a heavy retaining wall may be used. In addition to the moment caused by the internal pressure the slab will usually have to resist some direct bursting tension and have to withstand wind pressure when the bunker is empty. Otherwise the arrangement of main reinforcement may be the same as that in floor slabs (see Chapter XIII). If the vertical walls are called on to act as vertical girders, then vertical steel is required to carry the suspension tension and shear tension across the horizontal construction joints. This steel is generally difficult to calculate but vertical steel whose sectional area is 0·25% of the plan area of the wall slab is about the minimum.

If the wall is not acting as a vertical girder then horizontal cracks are most unlikely and only a few ⅜-in. diam. bars are required to support the horizontal steel. These may amount to as little as 0·1% of the concrete area.

Design of Counterforts.—External counterforts of uniform section may be reinforced as the T-beams in Chapter XIV. Internal tapering counterforts may be treated as the counterforts in Chapter XXXIII. The main steel in a uniform counterfort 20 ft. 0 ins. high tied in at the bottom and 5 ft. 0 ins. from the top is shown in Fig. 260 which also shows the loading.

Lining of Bunkers.—Coal bunkers need no lining provided the hoppers have sufficient slope. Hoppers sloping less than 40° may be lined with steel plates or glass tiles to reduce friction and aid

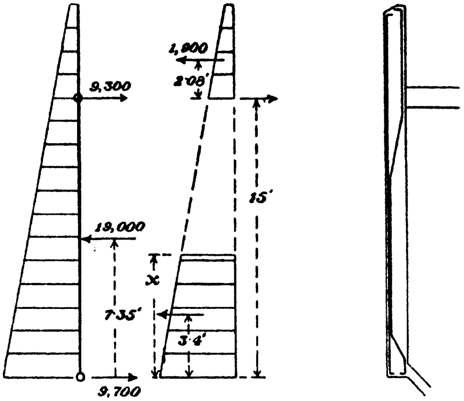


Fig. 260.

discharge. Coke has heavy abrasive powers. Hopper slopes are often covered with blue quarry tiles. The walls may be covered with similar tiles or may be made say ½-in. thicker with 1½-in. cover on the inside to allow for wear.

Bunker Columns.—The bending in the columns caused by wind may be found from Figs. 122, 123 and 128, the vertical spacing of horizontal wind braces being generally 15 ft. 0 ins. to 20 ft. 0 ins. If the corner columns are smaller than those under the middle of the structure it may be assumed that they take only half as much horizontal wind reaction each. The column bars must be well anchored against wind bending stresses at both top and bottom.

Quantities in Bunkers.—The steel factors for walls are fairly simple, but the steel factors for the beams and counterforts vary. A continuous horizontal beam without compression steel at mid-span will work out at a factor of about 18. A counterfort such as shown in Fig. 260 will have a factor of about 14. If this counterfort has four bars ⅞-in. (2 straight and 2 bent) the total steel per yard height of counterfort is about

$$2{\cdot}40 \times 14 = \text{say } 34 \text{ lbs. per yd.}$$

including light top bars and stirrups.

Ties will have a similar factor. A tie having four bars 1 in. diam. (one in each corner) would have about $3{\cdot}14 \times 14 = 44$ lbs. per yd. including light links.

The quantities of concrete and shuttering present no difficulties.

Bunkers.—Conclusion.—The beginner is not likely to hit upon the best arrangement and spacing of beams, counterforts and ties at the first attempt. When in doubt it is best to make a start on the first arrangement that comes to mind and work it out completely with quantities. A survey of the result will indicate clearly what modifications are required. Fig. 261 shows a few actual sections.

(For a collection of data on weights and angles of repose the reader is referred to *Walls, Bins and Grain Elevators* by Ketchum. The author cannot altogether agree with Professor Ketchum's calculated pressures.)

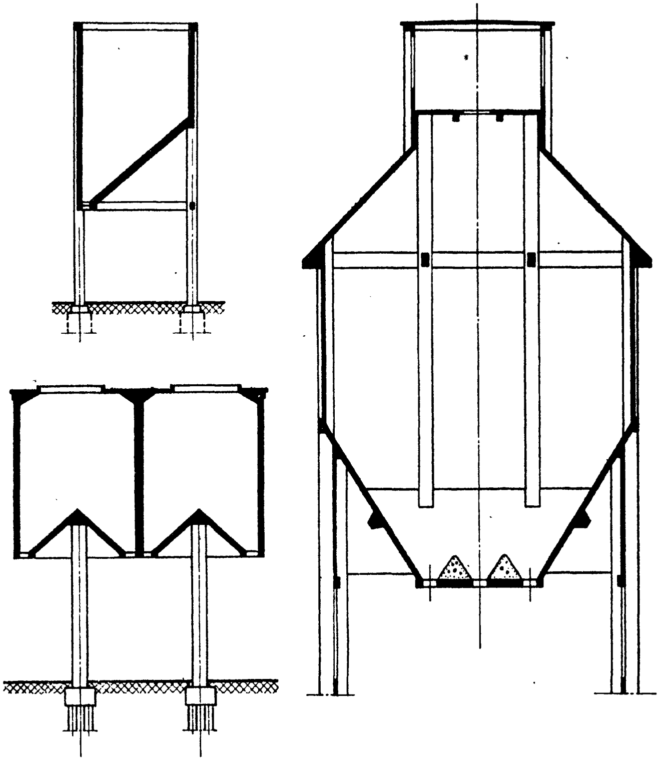


Fig. 261.

Silos.—There are two fairly well-known methods of calculating the pressures in deep bins and silos, one due to Janssen and one due to Airy. The latter is longer, somewhat inconsistent in its arguments, and offers no advantages over the former. The pressure in any granular mass depends on the exact conditions, particularly on the direction of movement or attempted movement. According to Rankine the relation between the intensities of pressure on two planes at right-angles lies between $\frac{1 + \sin \Phi}{1 - \sin \Phi}$ and $\frac{1 - \sin \Phi}{1 + \sin \Phi}$ and any ratio between these limits will hold the mass in equilibrium.

The argument which follows is based on Janssen's Theory.

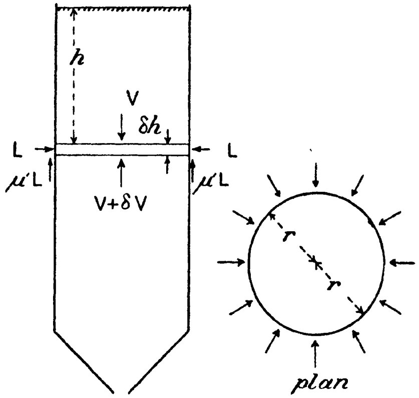


Fig. 262.

Fig. 262 shows a vertical section through a deep silo which is circular in section and is full of grain (say).

Let μ' = co-efficient of friction
w = weight of grain
V = vertical pressure per unit area
L = lateral pressure „ „ „
r = radius of bin
k = ratio of L to $V = \frac{L}{V}$.

First of all we must determine the conditions of movement or attempted movement. Such a silo is invariably filled from the top and emptied from the bottom. The contents, therefore, are always moving or tending to move downwards. The side walls merely resist what pressure comes on them and tend to fail by bursting outwards. We should therefore expect that the lateral pressures tended to their minimum value. It will be assumed that the intensity of vertical pressure is constant over a horizontal plane section and that the horizontal pressure is some constant fraction of the vertical pressure at all points.

(The admissibility of these two assumptions will be argued later)

$$\therefore\ k = \text{constant for any one case.}$$

Take a horizontal section through the silo at a depth h from the free surface and another at a depth of $h + \delta h$.

Consider the equilibrium of the portion of grain between the two horizontal sections. The grain above the top surface exerts a pressure V on the top and the grain below exerts a pressure $V + \delta V$. The sides of the bin exert a lateral pressure L and a frictional force $\mu' L$. Since the grain tends to move downwards, this frictional force is always upwards. The vertical pressures V are applied over an area of πr^2 while the reactions of the walls are applied over an area $2 \pi r \, \delta h$. The only other force is the self-weight of the grain which is $w \times \pi r^2 \times \delta h$.

Equating the vertical forces we have

$$\pi r^2 w \, \delta h = ((V + \delta V) - V) \times \pi r^2 + 2 \pi r \, \delta h \times \mu' L.$$

Now $\qquad L = k V$

$$\therefore \; \pi r^2 w \, \delta h = \pi r^2 \, \delta V + 2 \pi r \mu' \, \delta h \times k V$$

$$w r \, \delta h = r \, \delta V + 2 \mu' k V \, \delta h$$

$$\delta h \, (w r - 2 \mu' k V) = r \, \delta V$$

$$\text{i.e.,} \quad \frac{d h}{d V} = \frac{r}{(w r - 2 \mu' k V)}.$$

Integrating :—$h = \dfrac{r}{-2 \mu' k} \log (w r - 2 \mu' k V) + \text{constant}$

$V = 0$ when $h = 0$

$$\therefore \quad \text{constant} = + \frac{r}{2 \mu' k} \log (w r).$$

The logarithms being Naperian

$$L = k V = \frac{w r}{2 \mu'} \left(1 - e^{-\frac{2 \mu' k h}{r}} \right) \quad . \quad . \quad . \quad . \quad (1)$$

If h becomes large then $e^{-\frac{2 \mu' k h}{r}}$ tends to zero

and L becomes $\dfrac{w r}{2 \mu'}$.

The limiting value of L is therefore independent of the value of k.

In working out the foregoing we have made two assumptions. The first of these, that the vertical pressure is constant over a horizontal plane is probably not quite true, as the pressure is greater in the centre and is constant over the surface of a dome which is higher in the centre than at the sides, the grain supporting itself in a series of domes as coal supports itself over the opening in a bunker by arching action. The second, that the lateral pressure is some constant fraction of the vertical pressure, is probably in some way connected with the first assumption. The angle of repose of wheat is 28° and if the pressure were constant over horizontal planes then Rankine's conditions would be satisfied and k would be $\dfrac{1 - \sin 28^\circ}{1 + \sin 28^\circ}$ $= 0{\cdot}36$. Attempts to measure this value give results ranging from 0·3 to 0·67. The discrepancy seems to be due to the fact that the intensity of vertical pressure on a horizontal plane varies from $\dfrac{1}{0{\cdot}67}$ to $\dfrac{1}{0{\cdot}3}$ times the lateral pressure on the side, as we leave the side and get nearer the centre where the vertical pressure is largest. (The reader will find an excellent and exhaustive discussion on silo pressures in Ketchum's *Walls, Bins and Grain Elevators*.) It may be that the diameter of the bin has some bearing on the case, and that k is smaller for larger bins.

Example.—A silo is 10 ft. 0 ins. diameter and is filled with wheat at 50 lbs. per cub. ft. If $k = 0{\cdot}4$ and $\mu' = 0{\cdot}4$, what lateral pressures will be found at depths of 20 ft. 0 ins., 40 ft. 0 ins. and 60 ft. 0 ins.

The values of e^{-x} will be found in Dale's *Mathematical Tables* or some similar work.

$$\frac{w\,r}{2\,\mu'} = \frac{50 \times 5}{2 \times {\cdot}4} = 313$$

$$\frac{2\,\mu'\,k}{r} = \frac{2 \times {\cdot}4 \times {\cdot}4}{4} = {\cdot}064.$$

If $h = 20'\ 0''$, by equation (1)

$$L = 313\,(1 - e^{-1{\cdot}28})$$
$$= 313\,(1 - {\cdot}278) = 227 \text{ lbs. per sq. ft.}$$

if $h = 40'\ 0''$, $L = 313\,(1 - e^{-2{\cdot}56}) = 289$ lbs. per sq. ft.

If $h = 60'\ 0''$, $L = 313\,(1 - e^{-3{\cdot}84}) = 305$ lbs. per sq. ft.

If h tends to infinity L tends to 313.

The value of V is $\dfrac{L}{k}$ and tends to 780 lbs. per sq. ft. which is the weight of a column of grain 15·6 ft. high. In such a deep silo, therefore, the weight coming on the hopper bottom would be that due to about 16 feet of grain and all the rest would be supported by the frictional forces on the sides.

Extension to Square Silo.—Up to the present we have dealt with circular silos as the lateral pressure is obviously constant round the circumference. In a square silo the lateral pressure will be somewhat greater in the centre of the side than near the corners. If we neglect this, then the hydraulic mean depth which is the determining factor is $\dfrac{(\text{side})^2}{4 \times (\text{side})} = \dfrac{\text{side}}{4}$.

The hydraulic mean depth of a circle is $\dfrac{r}{2} = \dfrac{\text{diameter}}{4}$.

Therefore a square silo may be taken to have the same pressures as a circular silo whose diameter equals the side.

Practical Application.—What most concerns us in a deep silo is the maximum value of the lateral pressure and this approaches $\dfrac{w\,r}{2\,\mu'}$ where r is the radius (or half the side for a square silo). In determining the pressure on the hopper bottom the proper value of k must be used. As the correct value seems to be still a disputed point, a fairly low value should be chosen. A value of $\dfrac{1 - \sin \Phi}{1 + \sin \Phi}$ where Φ is the angle of repose should give results on the safe side. If there is any doubt about the value of μ' this may be taken somewhat low for safety. In checking the vertical loads carried by the sides, it must be remembered that assumptions giving an excessive value to the vertical pressure on the hopper bottom will give values of the load on the sides which are too small. Cases have been known where steel silos, when μ' had been taken as zero " for extra safety," have collapsed by crippling of the sides due to vertical thrust.

In Fig. 263 lateral pressures have been calculated for $w = 50$, $\mu' = 0{\cdot}4$, $k = 0{\cdot}4$ for different diameters.

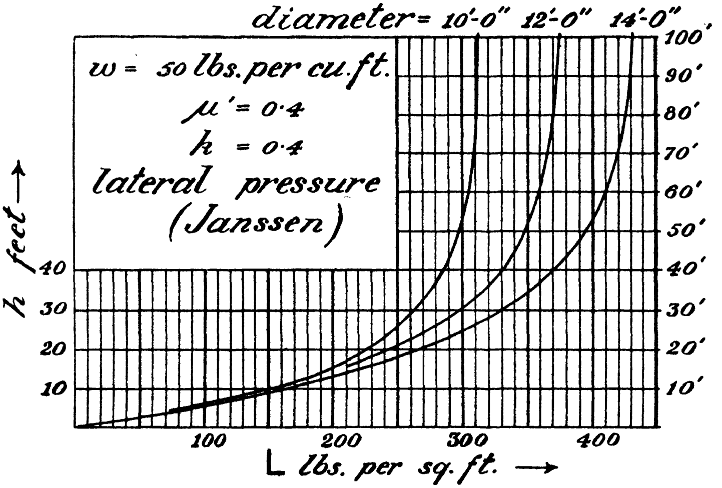


Fig. 263.

Shape and Size of Bins.—Isolated silos are usually circular. The shape and height of each individual bin in a battery of silos depends on the requirements of the millers and the area of site available. Sometimes the owners require a number of small bins of about 50 tons each (say 7 ft. 0 ins. × 7 ft. 0 ins. × 46 ft. 0 ins. high). Sometimes they deal in batches of say 300 tons (12 ft. 0 ins. × 12 ft. 0 ins. × 100 ft. 0 ins. high). In some cases of storage silos there is no need to divide the grain into batches and the very cheapest possible arrangement to store a given total volume of grain may be used. The maximum height of bin so far built in this country is about 100 ft. 0 ins. but there is no structural reason why bins of 120 ft. 0 ins. or 150 ft. 0 ins. depth should not be built.

In most batteries of silos in this country the bins are limited in size and are square (or very nearly so), although one battery at least has hexagonal bins. If there is any doubt about which size will prove cheapest, several must be tried. For example if the bins are each to hold 200 tons at 50 lbs. per cub. ft. then they might be 9 ft. 6 ins. × 9 ft. 6 ins. × 100 ft. 0 ins. high or 12 ft. 3 ins. × 12 ft. 3 ins. × 60 ft. 0 ins. high. Both arrangements (if the site permits) could be worked out and priced.

If there is no limit to the size of each bin the arrangement of large circular bins in Fig. 264 with the spaces between used as interspace bins will probably give the cheapest storage.

Design of Square Bins.—If we have a single rectangular bin whose sides are L_1 and L_2 subject to an internal pressure p the reversed moment at each corner is

$$\frac{p\,(L_1^2 - L_1 L_2 + L_2^2)}{12}$$

In the case of a square bin the moment is $\frac{pL^2}{12}$ at the corners and $\frac{pL^2}{24}$ in the centre of each side. In a battery of silos we must allow for the case when one bin is full and all the adjoining bins are empty and although the adjoining walls may slightly influence the moments in a rectangular bin we may work to the moments given above. Fig. 265 shows an arrangement which was often adopted in pre-war work. The arrangement follows closely that suggested by Professor Emil Mörsch but is not suited to modern methods of construction. A discussion of these is outside the scope of this book but a working knowledge of them is almost essential to efficient and economical design. Fig. 266 shows a bin similar in size to that in Fig. 265 but of more recent design. A difficulty arises about the exact length of span of the wall. Academically we might take a span of 12 ft. 6½ ins. Assuming that the heavy brackets fix the tangents at the corners we might take 10 ft. 6 ins. Against this the pressure in a square bin is

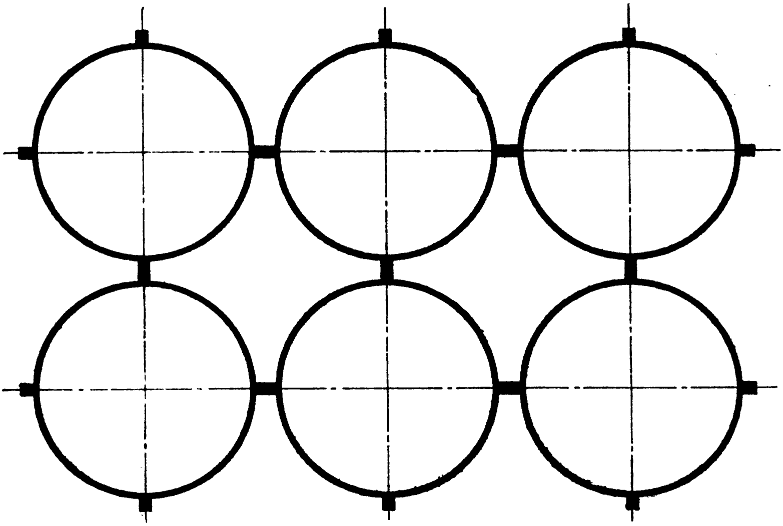


Fig. 264.

heavier at mid-span than elsewhere. If we assume that the effective span is equal to the clear width of 12 ft. 0 ins. we shall be on the safe side. The design of the wall in Fig. 266 to stresses of 750 and 18,000 is as follows ($\mu' = 0{\cdot}4$, $w = 50$ lbs. per cub. ft.).

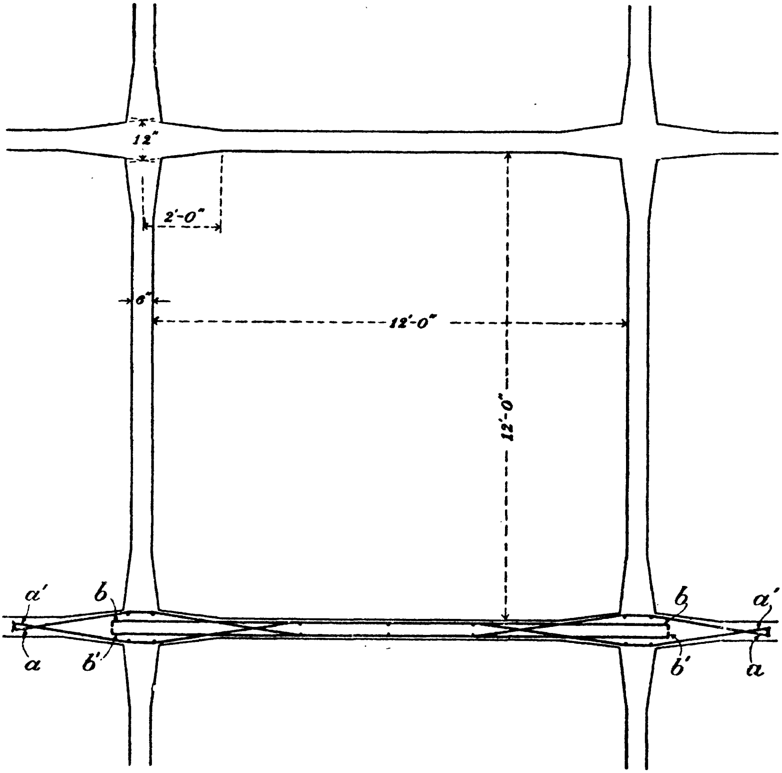


Fig. 265.

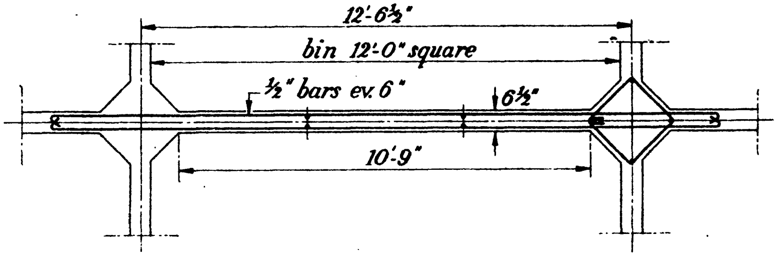


Fig. 266.

Hydraulic radius of bin = 6 ft. 0 ins.

Pressure at bottom approximates to

$$\frac{wr}{2\mu'} = \frac{50 \times 6}{2 \times 0{\cdot}4} = 375 \text{ lbs. per sq. ft.}$$

$$M \text{ at mid-span } = 375 \times 12{\cdot}0^2 \times \frac{12}{24} = 27{,}000 \text{ lbs.-ins.}$$

$$M \text{ at supports } = 375 \times 12{\cdot}0^2 \times \frac{12}{12} = 54{,}000 \text{ lbs.-ins.}$$

Direct tension = 375 × 6 = 2250 lbs.

Dividing the direct tension at mid-span equally between top and bottom steel and allowing ¾-in. cover we have at mid-span

$$A_s \text{ for } M = \frac{27{,}000}{18{,}000 \times 0{\cdot}86 \times 5{\cdot}5} = 0{\cdot}317 \text{ sq. ins.}$$

$$A_s \text{ for tension } = \frac{2250}{2 \times 18{,}000} = 0{\cdot}063 \text{ sq. ins.}$$

$$0{\cdot}380 \text{ sq. ins.}$$

Use ½-in. bars at 6-in. centres = 0·392 sq. ins.

As all bars overlap at all supports we have 0·784 sq. ins. of steel top and bottom at the support section.

$$M = 54{,}000 \text{ lbs.-ins.}$$
$$\text{tension} = 2250 \text{ lbs.}$$

Transfer the tension 2¼ ins. to the line of the tensile steel (see Chapter VII).

$$\text{Modified } M = 54{,}000 - (2250 \times 2{\cdot}25'')$$
$$= 48{,}900 \text{ lbs.-ins}$$

$$R = \frac{48{,}900}{12 \times 5{\cdot}5^2} = 135 \text{ (quite safe)}$$

$$A_s \text{ for } M = \frac{48{,}900}{18{,}000 \times 0{\cdot}86 \times 5{\cdot}5} = 0{\cdot}572 \text{ sq. ins.}$$

$$A_s \text{ for tension } = \frac{2250}{18{,}000} = 0{\cdot}125 \text{ sq. ins.}$$

$$0{\cdot}697 \text{ sq. ins.}$$

We have actually *0·784 sq. ins.*

Design of Circular Silos.—The design of the circular bins with a hoop tension of say 16,000 is clear from Chapter XX. The design of the interspace bins in Fig. 264 is more difficult. The curved sides could, of course, act as arches but there is really no support for them and it is safer to arrange the reinforcement in the walls so that they can span. The hydraulic radius of the interspace bins is much smaller than that for the circular bins and the pressure is correspondingly less. Rearrangement and some additional overlapping of the hoop steel is generally sufficient to hold the interspace bin when the surrounding circular bins are empty.

Vertical Loads and Bottoms.—The vertical friction brings heavy vertical loads on the walls and care is needed to transfer this to the columns under the bins. The detail at the junction of column and hopper bottom is often an awkward one. Complicated hopper bottoms are usually built with a thick horizontal slab, the slopes being put in afterwards in mass concrete. In many modern silos the hopper bottoms are wholly or partly built of steel plate.

The design of regular reinforced concrete hopper bottoms is clear from Chapter XXII.

Silos Conclusion.—Little trouble need be anticipated from the inside walls. The outside walls must be weathertight and may be split by shrinkage cracks, particularly the lower part of the walls. Shifting of the reinforcement is sometimes difficult to prevent and rusting and blowing of the outside steel may result. To remedy these possible sources of trouble a good band of steel in the lower part of the outside walls and 1 in. to 1½-in. cover to the external steel should prove effective. Some designers still cling to a high value of μ' forgetting that μ' in modern silos gets smaller with the passage of time as the walls wear smoother.

It is standard practice to keep the same section of concrete from top to bottom of the walls although the top twenty feet or so is occasionally reduced in thickness.

Cement Silos.—During the last few years the author has been called on to design circular cement silos up to 140 feet high with capacities ranging up to 5,000 tons per compartment. There seems to be little available data on which to base a design but the author takes $w = 90$, $\mu' = 0{\cdot}4$ and $k = 0{\cdot}42$ (Ketchum).

Sliding Shutters.—Most silos over eighty feet high are now built by continuously moving shutters. In silo work we have reached the point where the requirements of the method of construction generally take precedence over academic considerations of theoretical design and a designer will often be unable to produce the most economical design for a battery of silos unless he has a working knowledge of the main principles of sliding formwork.

CHAPTER XXXII

BRIDGES

A bridge consists of a single suspended floor. Unlike the design of floors in buildings the question of bridge floors is often settled by the cost of the supports. In a warehouse floor which is supported on short columns constructed under simple conditions, the larger the number of columns the cheaper the floor, as the cost of the columns is a comparatively small item. This is not the case in bridge design where it is necessary to strike a balance between the cost of foundations and the cost of the superstructure in order to arrive at the most economical design.

Loading.—As the loading is now always fixed by some authority, there is little point in discussing the matter. Even when loads are definitely specified there is no harm in examining the specified loading critically. Where the specified load appears unjustifiably large the smallest possible sections that will suffice should be employed. Where the specified load appears small, ample provision should be made. Failure of a structure through any cause whatever is bound to lower the prestige of the designer in the eyes of those who do not know all the details. Where the load is applied at the surface of earth filling, a " spread " of 30° to the vertical may be assumed. (Figs. 115 and 173.)

Types of Bridge.—Single span bridges may be any of the following types :

(1) Slab.
(2) Slab-and-girder.
(3) Arched (including square-arched) and tied arches.
(4) Framed girder.
(5) Double cantilever (continuous).

Intermediate supports are usually trestles or piles.

(1) For small spans a bridge may be built as a large box culvert the bridge floor and abutments consisting of plain slabs. It may or may not be advisable to design the whole as a square arch depending on the height of the abutments, etc.

For spans of 30 ft. 0 ins. or 40 ft. 0 ins. the deck may consist of a simple thick slab (see the author's description in *Concrete*, Jan., 1933).

(2) The slab-and-girder type is very popular. It appears from existing structures that a 6-in. floor-slab will safely resist the punching shear from the heaviest highway wheel load and this thickness may be looked on as a minimum. Highway bridges very seldom

have more than two systems of beams, main beams running in the direction of the bridge and secondary beams running across. Railway bridges may have stringers under the rails in addition.

For all spacings of cross girders which are less than the distance of axle to axle, the maximum moments and shears will occur when the heaviest axle (or axles if the cross girder is long enough to hold two vehicles side by side) occurs over the girder in question. However close together the cross girders are spaced they will still have to take the same rolling load. Diminishing the spacing of the cross girders will only reduce the total moment on them by a small amount of dead load stress. The slab should therefore be spanned to its uttermost (up to say 10 ft. 0 ins. as an upper limit). The cross girders should be cantilevered past the outside main beams. This will reduce their span and make the moments on the main beams more even. Where the bridge consists of more than one span the most economical spacing of supports must be found by trial and error. As a preliminary guide the span may be chosen as equal to the height of the highest trestle (this does not apply to pile trestles). For pile trestles the span will be limited to some extent by the safe load carried by each pile and by the total number of piles, as the cost of driving per pile diminishes with the total number driven. For narrow bridges where headroom is very scarce parapet girders may be used. Some authorities (quite rightly) forbid their use. These are frequently designed on the steel-beam theory which method is quite safe in the hands of an expert who knows what limits he may go to. The beginner should work to reasonably low stresses in the steel top and bottom until he knows what latitude the ratio of span to depth of girder, will give him. (See Chapter XXXV.)

(3) True arches should never be adopted unless the foundations are above suspicion. The application of the various shapes of curved arch has already been discussed in Chapter XVIII.* The great point in their favour seems to be their appearance which most people prefer to that of a girder bridge. They also give a very small depth of construction at the crown. For a single span bridge with reinforced concrete abutments the slab-and-girder type may be merged into a square arch by spacing vertical counterforts in the abutments to line with the main girders in the bridge floor. The square corners should be rounded out by splays as much as possible and great care must be taken in designing them. (See Chapter XIX.)

(4) Main girders of long span may be lattice girders, either *N*, bow-string or hog's-back in shape. They should invariably be completely triangulated and calculated as pin-jointed structures. An eminent Belgian engineer informed the author of a case of a large framed girder without diagonals where the necessary steel arrangement at the junctions of the members (compare the steel arrangement at the brace junctions in Chapter XIII) was so complicated that the concretors put the concrete in in handfuls and poked it round the steel with their fingers. Apart from the doubtful

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* See the author's *Reinforced Concrete Arch Design*.

strength of such a " monolithic " joint the extra labour costs would have paid for the provision of the diagonals, necessary to triangulate the structure fully, many times over. The author considers that incomplete framed girders, although very interesting from an academic standpoint, are not economical except perhaps for very light loads. The design of framed girders follows the lines adopted for similar steel girders, the most important point being the junctions between the members. (See Chapter XV.)

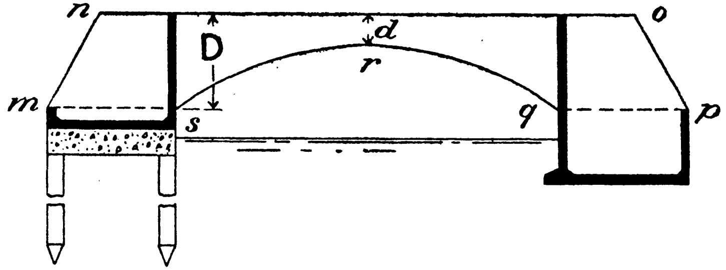


Fig. 267.

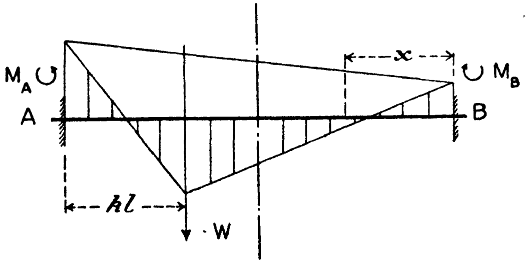


Fig. 268.

(5) This type of bridge is an exaggerated square arch. It may be made in the outline of an arch spandrel but puts no horizontal thrust on the abutments. Owing to the large depths which may be used at the abutments the ordinary theory of beams does not apply fully, but will be used to give some idea of the moments.

The members consist of encastré beams with a variable moment of inertia the variation throwing nearly all the bending moment on the abutment sections, producing a double cantilever effect.

Suppose Fig. 267 shows an encastré beam of constant width but whose depth varies, the soffite being approximately parabolic. The beam is symmetrical the depth varying from d at mid-span to D at the abutment. If we assume that I at every section is proportional to the (overall depth)^3 we can calculate the fixing moments M_A and M_B for any position of the load W. The moments are shown in diagram in Fig. 268, and a complete series of values for different ratios of D to d are given in Table CC I. (Page 462.)

In practice the question is generally complicated by the fact that the beam has a T-section at mid-span and a rectangular section at the support and it may be necessary to summate the slopes and deflections from first principles by some semi-graphical method. In any such case the designer may calculate the value of I at mid-span and at support and take the value of the cube root of the ratio between them. The moments in the actual bridge should be close to those in Table CC I taking a value of $\frac{D}{d}$ equal to this cube root. If in an actual bridge the moment of inertia at the support is 30 times the moment of inertia at mid-span this will correspond closely to the case in Table CC I where $\frac{D}{d} = \sqrt[3]{30} = 3{\cdot}1$.

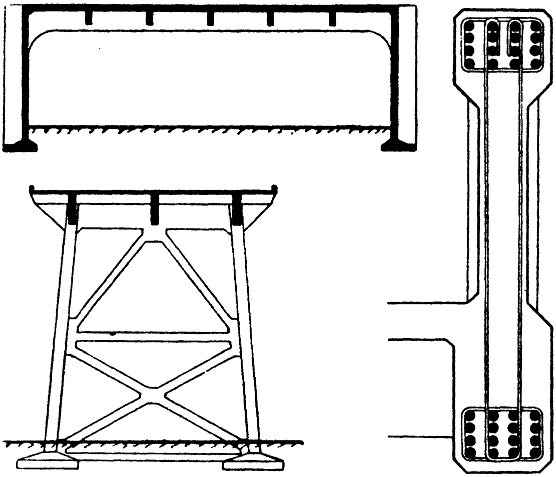


Fig. 269.

In applying the results we must remember that it is impossible to fix the ends of the beam absolutely, particularly if the abutments are carried on long piles, and additional strength near mid-span should always be provided above the calculated values.

Abutments.—As a rule the abutment has to combine the functions of a support and a retaining wall and may most easily be arranged by spanning the slab horizontally and spanning the counterforts vertically. The tops of the counterforts are supported by the deck structure of the bridge. In the wing-walls the counterforts (if any) must cantilever.

The abutments for arch bridges are best constructed in solid concrete. They may be made of mass concrete with a few bars in, or constructed as reinforced concrete cells with thin walls and afterwards filled up with poor concrete helped out by displacers, etc.

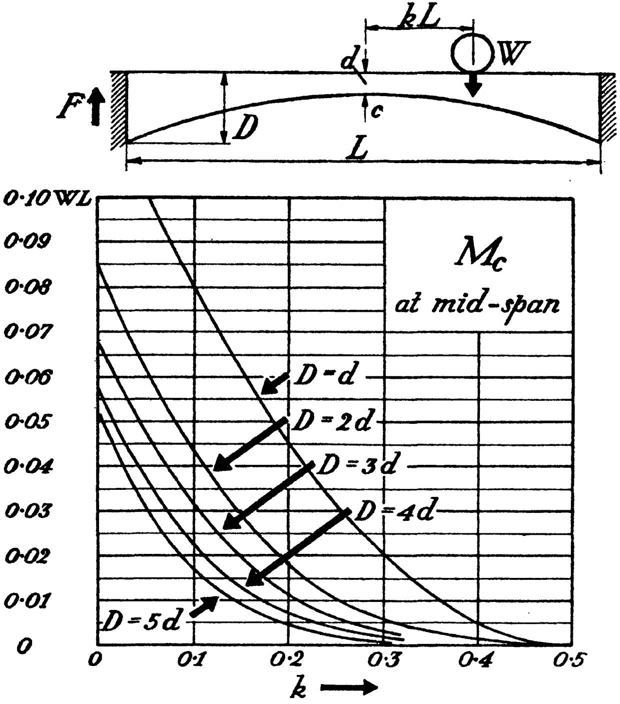
The horizontal thrust should be covered by frictional resistance between the foot of the abutment and the foundation.

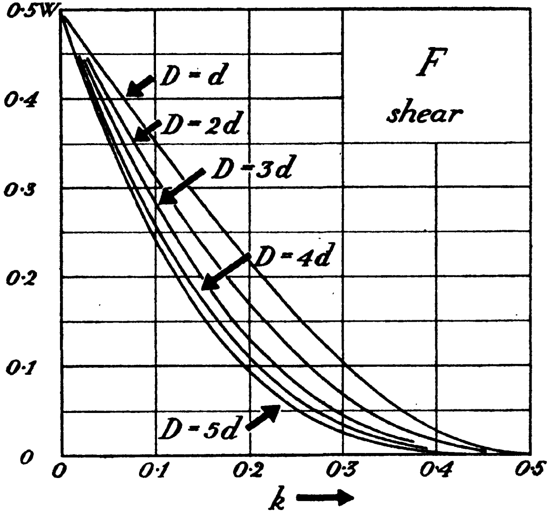
The design of deep or difficult foundations is largely a matter of understanding the different processes of contracting—cofferdams of different types, compressed air work, heavy timbering, open caisson work, pumping, consolidation processes, etc. It is, again, of little use discussing any of these without going into the detailed cost of each operation. This would need a separate book.

Fig. 269 (based on existing work) shows a longitudinal section through a square arch, a cross-section through a trestle bridge and a detail at mid-span of a parapet girder. In modern design the cross girders would be omitted and the diagonal braces would be replaced with vertical and horizontal members. As previously mentioned, parapet girders are generally forbidden in modern design.

TABLE C.C.I

SYMMETRICAL ENCASTRÉ BEAMS
PARABOLIC SOFFITE
I VARIES AS $(DEPTH)^3$





CHAPTER XXXIII

RETAINING WALLS

Types.—Whenever possible, as in bridge abutments or deep areas round buildings, the wall may find lateral support on other parts of the structure. Isolated walls must cantilever. The main types

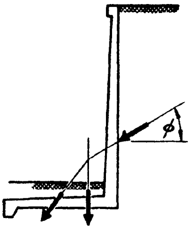


Fig. 270.

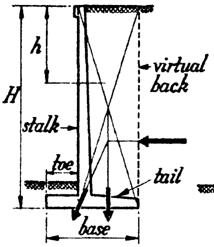


Fig. 271.

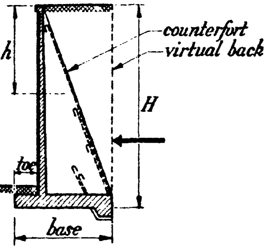


Fig. 272.

of wall in general use are shown in Figs. 270, 271 and 272. In Fig. 270 the earth pressure comes on the actual back of the concrete and may be assumed to act at an angle ϕ. In Figs. 271 and 272 the pressure comes on the virtual back of the wall and must be assumed to act horizontally.

Pressure.—The lateral pressure from all materials except clay is computed as $\frac{1}{2}\ k\ H^2$ where k is an equivalent fluid pressure varying from 10 to 35 lbs. per cub. ft. according to the type of filling, height of original ground line, etc. The pressures from clays are given at the end of this chapter.

Stability.—Stability failures are usually confined to walls on clay foundations and the history of the failure repeats itself with monotonous regularity. A trench is opened for the footings and discloses a bed of firm clay. The wall is built and the trench filled back with loose material. This makes an effective drain and all surface water collects in the trench. Under the continual soaking the clay softens and squeezes out under the toe. The wall sinks, runs forward and tilts. To prevent this, water must definitely be excluded from the foundations, or walls on clay sites must be designed for very small foundation pressures. In theory a wall may overturn bodily about the extreme point of the toe or may slide forward on an even keel, but no practical wall ever fails in either of these ways. The coefficient of friction between the concrete base and the earth foundation is never less than 1·0 (that is 0·67 with a factor of safety of 1·5).

Proportions.—Stability failures are comparatively frequent while structural failure is practically unknown. The designer's first effort should be to find a design with a small bearing pressure on the foundation, unless the wall is on sound ballast or rock. As a general rule the base in Figs. 271 and 272 may be made as follows:

Base $= H \times \sqrt{\frac{3k}{(4\,w)}}$ where w is the weight of the filling behind the wall. In Fig. 271 one-third of the base may project in front of the wall to form a toe and in Fig. 272 one quarter.

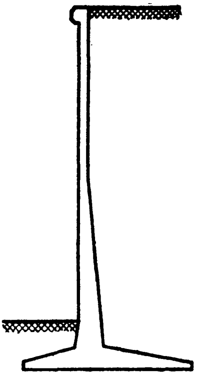


Fig. 273.

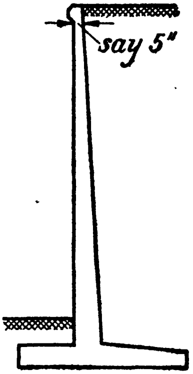


Fig. 274.

If the wall in Fig. 271 is 15 ft. 0 ins. high subject to a lateral pressure of 30 lbs. per cub. ft. while the backing weighs 100 lbs. per cub. ft.

$$\text{Base} = 15{\cdot}0 \times \sqrt{\frac{3 \times 30}{(400)}} = 7{\cdot}13 \text{ say 7 ft. 2 ins.}$$

Of this 2 ft. 5 ins. should form the toe.

The reader may draw curves showing the relation between height and base with different proportions of toe and different values of k (see the author in *Civil Engineering*, June 1933).

Slab Cantilever Walls.—For walls up to about 15 ft. 0 ins. high, the slab cantilever wall in Fig. 271 is cheapest. The stalk cantilevers and the moment at any section is proportional to h^3. The minimum effective depth required is proportional to $\sqrt{h^3}$. The toe cantilevers upward while the tail cantilevers downwards. The moment at the base of the stalk equals the sum of the moments in the toe and the tail. Some theoretical designers make the slabs close to their minimum theoretical thicknesses as in Fig. 273 but in practice it is neater and usually cheaper to keep to the simpler arrangement in Fig. 274. Having decided on the proportions, the designer calculates the moments in the stalk, tail and toe and the shear in the toe. From these the effective depths required are soon found and a section of the wall as in Fig. 273 or Fig. 274 is drawn. By taking horizontal sections through the stalk at 2 ft. 0 ins., 4 ft. 0 ins., 6 ft. 0 ins., 8 ft. 0 ins., etc., below top ground level the area of steel required may be drawn out as a graph and the steel arrangement is then easily fitted to it. If the area of main steel at the base of the stalk in Fig. 274 is A_s sq. ins. per ft. run of wall the weight of main steel in the wall is 30 A_s × (total area of stalk and base in sq. yds.) pounds. To this must be added the steel in the front of the wall and the horizontal steel.

Counterfort Cantilever Walls.—The standard type of wall is shown in Fig. 272. The bottom 3 ft. 0 ins. or so of the front slab cantilevers off the base while the remainder spans horizontally from counterfort to counterfort. The theoretical effective depth of the slab varies as $\sqrt{h}$. The earth pressure tends to tear the slab away from the counterfort. The counterfort is a T-beam of tapering section, the horizontal component of the tension in the steel taking part of the horizontal shear. The bottom slab spans from back beam to front slab and cantilevers out in front. The back beam spans from counterfort to counterfort, the joint between them requiring careful design. In most walls of this type the counterforts are spaced about half the height of the wall, but as a variation they may be spaced at a distance equal to the height of the wall, the front slab being designed as a mesh panel supported on a top beam. The method of treatment will follow Fig. 249.

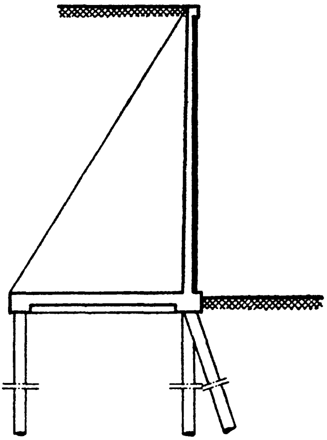


Fig. 275.

Retaining Walls on Piles.—The stability of a wall on piles as in Fig. 275 will be clear from the latter half of Chapter XXI. Increasing the rake of the front piles from 1 in 3 to 1 in 2 (if possible) will materially reduce the width of base.

HORIZONTAL PRESSURES ON VERTICAL WALLS, ETC.

MATERIAL	Weight lbs. per cubic foot	Angle of Surcharge	Coefficient of Friction		Equivalent Fluid lbs. per cubic foot	AUTHORITY
			Material on Material μ	Material on Concrete μ'		
Sand (loose dry) . .	94	0°	0·90		18	Crosthwaite (measured)
do. . .		+ 30°			28	,, ,,
do. . . .		− 30°			13·8	,, ,,
Sand (compacted) .	104	0°	1·15		15	,, ,,
Sand (loose shovelled)	81	0°	0·70		17	Fulton ,,
do. .		+ 35°			37	,, ,,
Gravel (loose shovelled)	84	0°	0·60		20	,, ,,
do. .		+ 31°			42	,, ,,
Earth (loose shovelled)	82	0°	0·87		14·4	,, ,,
do. .		+ 41°			26	,, ,,
Bituminous Coal .	50	0°		·325	11·7—13·5	Ketchum (calculated)
Anthracite Coal . .	52	0°		·287	16·8—18·6	,, ,,
Sand	90	0°		·325	21·8—25·5	,, ,,
Ashes	40	0°		·60	6·0— 8·7	,, ,,

The two values given by Ketchum are for rough and smooth walls respectively.

Bell's Formulae for Clay.—(Level Surface). Intensity of active pressure on back of wall at a depth h

$$= w\,h \tan^2\left(\frac{\pi}{4} - \frac{a}{2}\right) - 2\,k \tan\left(\frac{\pi}{4} - \frac{a}{2}\right).$$

Maximum value of counterpressure at depth h

$$= w\,h \tan^2\left(\frac{\pi}{4} + \frac{a}{2}\right) + 2\,k \tan\left(\frac{\pi}{4} + \frac{a}{2}\right).$$

Minimum allowable intensity of downward pressure on foundation at heel of wall at depth h

$$= w\,h \tan^4\left(\frac{\pi}{4} - \frac{a}{2}\right) - 2\,k \tan^2\left(\frac{\pi}{4} - \frac{a}{2}\right) + 2\,k \tan\left(\frac{\pi}{4} - \frac{a}{2}\right).$$

Maximum allowable intensity of pressure on foundation at toe of wall at depth h

$$= w\,h \tan^4\left(\frac{\pi}{4} + \frac{a}{2}\right) + 2\,k \tan^3\left(\frac{\pi}{4} + \frac{a}{2}\right) + 2\,k \tan\left(\frac{\pi}{4} + \frac{a}{2}\right).$$

MATERIAL	Initial Shear Strength k lbs. per sq. foot	Increase in shear a	Consistency Impression		μ clay on clay
			Drop 12″	Drop 24″	
1. Very soft puddle clay . . .	450	0°	1·3″	1·5″	0
2. Soft puddle clay .	670	3°	1·1″	1·3″	0·05
3. Moderately firm clay	1120	5°	1·0″	1·2″	0·09
4. Stiff clay . . .	1570	7°	0·9″	1·0″	0·12
5. Very stiff boulder clay	3600	16°	0·5″	0·7″	0·27

The consistencies were standardized by dropping a steel ball $1\frac{5}{8}$ ins. diameter weighing 0·632 lbs. upon a sample slab of clay 3 ins. diameter and $1\frac{3}{4}$ ins. thick and measuring the diameter of the impressions produced.

Material	Weight per cubic foot	μ	μ' on concrete	Janssen's k	Authority
Wheat .	49	0·466	0·444		Airy
,, .	49	0·532	0·41	0·6	Jamieson
,, .				0·3	Pleissner
,, .	50			0·4	Ketchum
,, .				0·67	Janssen
Barley .	39	0·507	0·452		Airy
Oats . .	28	0·532	0·466		Airy
Maize . .	44	0·521	0·423		Airy
Beans . .	46	0·616	0·442		Airy
Peas . .	50	0·472	0·296		Airy
Tares . .	49	0·554	0·394		Airy
Linseed .	41	0·456	0·414		Airy

The data are taken from Crosthwaites' papers before the Inst. of Civil Engineers (Min. Proc. vol. CCIX (1920) and vol. CCIII (1917)). Fulton's paper in vol. CCIX (1920). A. L. Bell's paper vol. CXCIX (1915). Airy vol. CXXXI (1897) and from Ketchum's *Walls, Bins and Grain Elevators.*

The reader should refer to the originals.

CHAPTER XXXIV

CHIMNEYS

Although reinforced concrete is eminently suited to resist the structural stresses developed in high chimneys, many early chimneys had no proper insulation and no reinforcement to resist the high stresses caused by the temperature gradient through the shell. Improvements in insulation, provision of temperature reinforcement and a tendency to cooler flue gases have put reinforced concrete chimneys back into the commercial field particularly for stacks of over a hundred feet.

Structural Stresses.—A careful estimate of the self-weight and wind moment will give the stresses on a horizontal section by application of Tables HC I and HC II (pages 112 and 113). Allowance must be made for the additive temperature stresses. A wind pressure of 20 lbs. per sq. ft. of elevation is a usual allowance for circular shafts.

Temperature Gradient.—If a chimney has an independent brick lining with an air space as in Fig. 275a there being a total

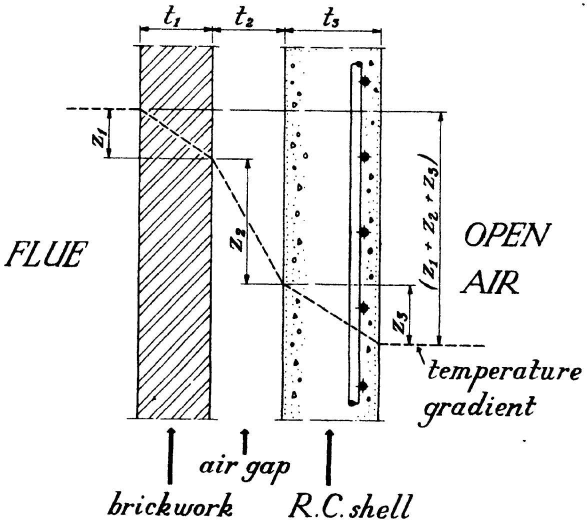


Fig. 275a

difference in temperature between the flue gases in contact with the brick lining, and the open air in contact with the outside of the shell of $(z_1 + z_2 + z_3)$ the drop through the concrete shell is z_3

$$z_3 \left(1 + \frac{c_3\ t_1}{t_3\ c_1} + \frac{c_3\ t_2}{t_3\ c_2}\right) = (z_1 + z_2 + z_3)$$

where c_1, c_2 and c_3 are the relative thermal conductivities and t_1, t_2 and t_3 are the relative thicknesses.

The relative thermal conductivities for reinforced concrete, brick, ventilated air space, unventilated air space, and möler brick are 10, 7, 4, 8 and 1·36 respectively. If the flue gases are at 500° F. while the air is at 0° F. and the chimney has a 6 in. reinforced concrete shell lined with 4½ ins. of brick with a 4½ ins. ventilated air space

$$z_3 \left(1 + \frac{10 \times 4{\cdot}5}{6 \times 8} + \frac{10 \times 4{\cdot}5}{6 \times 4}\right) = 500$$

$$z_3 = 131^\circ\ F$$

Vertical Temperature Stresses.—In a homogeneous chimney shell of plain concrete where the fall in temperature from the inside face to the outside face is 100° F. with a coefficient of expansion of 0·000006 and an elastic modulus of 2,000,000 lbs. per sq. in., it is clear that the stress will vary uniformly from 600 lbs. per sq. in. compression on the inner face to 600 lbs. per sq. in. tension on the outer face. These conditions will apply for the body of the chimney but there will be bending moments at the extreme base where the shaft joins the foundation, and at the extreme top where the vertical compression and vertical tension must vanish.

At the top the chimney tends to " bell " out and cause additional ring tension at the extreme top. This effect is very intense but dies away very rapidly. With the values stated above, the calculated ring tension at the extreme top is 200 $\sqrt{3}$ lbs. per sq. in. This ring tension dies away to zero at a distance of $0{\cdot}6\sqrt{rt}$ where r is the radius of the chimney in feet and t is the thickness of the shell in feet. For example, with a chimney 20 ft. diameter and 6 ins. thick, the effect would disappear 1 ft.-4 ins. below the top.

Circumferential Temperature Stresses.—Providing the shell is thin compared with the radius the circumferential stresses will be very nearly the same as the vertical stresses, that is plus or minus 600 lbs. per sq. in. for a difference of 100° F.

Taking the total thickness of the chimney and assuming that the steel takes all the tension and is stressed to 18,000 lbs. per sq. in., the ring steel should be 0·83% of the concrete area. Taking the effective depth only and assuming that this is three-quarters of the total thickness, the temperature range from the inside face to the steel will be only 75° and the area of steel is reduced to 0·47% of the total concrete area. The figure of 0·6% recommended by Suensen seems a good working value for an average shaft with independent brick lining .

For clarity the author has put the figures above in their simplest form. The reader may evolve more complicated formulae for himself but these are very apt to be misleading.

Details.—Openings near the base of the chimney should be made as narrow as possible and the main vertical bars should be swept to either side and taken down past the opening. The sides must be thickened out to compensate for the area of concrete cut out by the opening. The vertical steel being in beam tension must be lapped 45 diameters and hooked at the ends.

The deflection of chimneys may be computed by the method given for straight-cambered struts in Chapter XVI.

One great point in favour of reinforced concrete is the lightness of construction and the large area over which the load may be spread by a raft footing.

Great care is needed in arranging the details of the brackets (if any) that carry the brick lining and the details of the cap at the extreme top but these are not questions of reinforced concrete design.

Several firms specialize in chimney construction and it is as well to get in touch with them as special methods of construction reduce the cost to a considerable extent.

Bases to chimneys are usually circular thick slabs of reinforced concrete and present no special feature.

Local Ovality of Chimney.—If Fig. 275b is the section of a thin circular chimney of constant section subject to a wind pressure of p uniformly spread over the elevation, the wind will tend to distort the chimney locally as shown by the dotted line the moments being :—

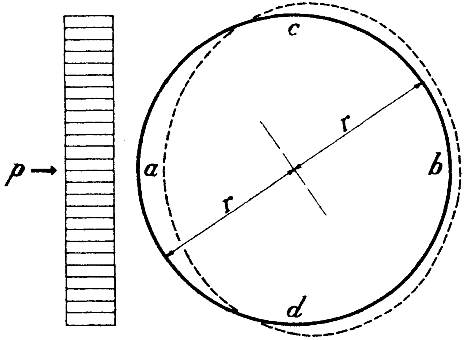


Fig. 275b

$$M_a = +\ 0{\cdot}13\ pr^2$$
$$M_c = M_d = -\ 0{\cdot}125\ pr^2$$
$$M_b = +\ 0{\cdot}12\ pr^2.$$

Periodicity.—A strong wind may blow in periodic gusts. If the natural period of vibration of the shaft coincides with the period of the gusts the chimney will fall. The period of the gusts is never less than three seconds and probably never less than six seconds.

Reference.—Some interesting measurements on a reinforced concrete chimney are given in the Proceedings of the American Concrete Institute 1926 and 1927.

CHAPTER XXXV

FIRST PRINCIPLES

This chapter is placed last as it represents the author's own opinions. In all previous chapters an effort has been made to put forward the principles of design as they are generally known and practised. These principles represent a tremendous volume of collected experience and any future changes which take place will be very slow and gradual. On the other hand it must be admitted that our knowledge of the subject is by no means complete and changes are gradually taking place and will take place for very many years.

Standard Design.—It is difficult to strike a line between absolute rashness on the one hand and timid conservatism on the other. The subject is further complicated by the fact that all Engineers worthy of the name have ideas of their own and while one employs high stresses in his beams and low stresses in his columns, another will adopt an opposite policy, etc. Stresses of 600 and 16,000 were recommended by the L.C.C. and R.I.B.A. and were in general use in the period 1913-1933. For building work stresses of 750 and 18,000 are now more common. In labelling good average design as " standard " design this fact has been borne in mind. No reflection whatever is meant to be cast on competitive designers who employ higher working stresses. In fact, as economy is one of the first duties of the designer, competitive work may be looked upon as the highest development of our knowledge. It is not too much to say that the whole of our present-day outlook on reinforced concrete, and its widespread use are entirely due to firms who have specialized in the preparation of competitive designs. In this connection it is impossible to exclude a mention of the Hennebique system both on account of the pioneer work that it accomplished and the tremendous number of large and important structures built in accordance with its designs. The originators of this system seem to have had a much clearer view of the underlying principles of design than many more modern " experts."

Object of Design.—When we talk of " designing " a structure what do we mean? What is our ultimate aim and why do we apply this process or that process? In all works on the subject constant reference is made to working stresses. The effects of working stresses and strains are quite invisible and have (as far as the owner of the structure is concerned) no tangible value. When a modern owner requires a structure, what are the essential points he looks for and pays for? The structure must be of the right size and shape, must be cheap and must carry the prescribed loads with a reasonable margin of strength. It is this last condition which is so difficult to express in exact terms. The owner of the building or other structure requires that he himself shall be quite satisfied that failure of the structure under working loads is impossible and requires that his mind shall be quite free from anxiety on the subject. It is clear that the strength in question is the ultimate strength of the structure, i.e., the greatest load the structure carries immediately before it collapses and fails to function further as a structure. In order to cover all human errors and variations the working load must be restricted to some fraction of the ultimate load. The reciprocal of this fraction is the factor of safety. If the ruling consideration is the ultimate strength, then why do we use methods based on working stresses, and are such methods correct. The answer is that such methods are unsound in principle, that they are founded on the conditions obtaining in a perfect elastic structure and are only approximately correct after much adjustment in practical design. Reinforced concrete under ultimate loads departs almost entirely from the conditions of a perfect elastic structure. This fact has been established by the long process of trial and error known as standard practice. The theory of the subject, which always follows later, has not been fully evolved. Various theories based on ultimate loads have been developed in America, notably by Professor Talbot* but these have not been adopted in this country and the author has grave doubts as to whether they would apply to practical beams incorporated in an actual structure, as the conditions of support affect the problem, particularly the ultimate stress in the tensile steel at mid-span. Moreover the author would go further. Not only must sections of members be designed for ultimate strengths, but the moments, etc., at all sections must be those which would occur immediately before disintegration. In short, the working load must be based on the ultimate strength of the structure as a whole. This is the true basis and object of design.

Foundation of Methods of Design.—In deciding what ultimate strength may be expected from a structure, on what shall we base our arguments? The following sources of information should be referred to in the order named.

(1) Extensive and exhaustive tests on actual practical structures built under normal commercial conditions: such tests to be tests to destruction.

(2) Standard Practice, particularly as exploited in keen competition.

(3) Extensive tests on commercial buildings under working loads.

(4) Tests on fairly large isolated specimens in testing machines.

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* *Principles of Reinforced Concrete Construction* by Turneaure and Maurer.

(5) Tests on models and small pieces and other so-called " research work " of similar type.

(6) Arithmetical analysis founded on assumptions of perfect elasticity.

To discuss these sources of information seriatim.

(1) Such a series of tests, owing to the cost and risk, is not likely to be carried out. It may be possible to test parts of disused structures which are about to be demolished and the evidence from such tests would prove extremely valuable. (Univ. Ill., Bull. 106.)

(2) This represents by far the largest and most reliable source of information at present available. Standard practice represents an infinite number of trial and error designs. Competitive designers have in course of time grown bolder and bolder and cut their designs finer and finer. This process takes a long time to mature and the gradual inroads upon the factor of safety are made very slowly. It will be seen that standard practice automatically adjusts itself to ultimate strengths as indicated by actual failures. This explains why methods of design based on working stresses must be varied for different kinds of members carrying different sorts of stress.

(3) Extensometer tests on commercial structures under working load are very useful in indicating the nature and distribution of stress and in showing the great discrepancy between measured results and results calculated from the simple elastic theory. Ultimate strengths may be guessed at by extrapolation. Such a process is a bit risky but is preferable to having no evidence whatever to work on.

(4) Tests in laboratories do not reproduce the end conditions which play such a large part in practice. Specimens so tested should be concreted by skilled workmen and not by students.

(5) Tests on models, etc., are useful as indicating the nature of the moments and shears for students, etc. Attempts to measure the stresses in small models and then apply the results to full-size structures is an extremely dangerous practice.

The author does not know of any single value or method used in practical design which has been found by or deduced from any of the research work carried out in this country. These remarks do not apply to work done in America where the size of the tests and the sensible outlook of the commentators form a striking and most unhappy contrast to " research work " done here.

(6) Academic analysis should never be looked upon as independent evidence. It affords, in some instances, a fairly simple and reliable explanation of results already established by practice. Where there is any discrepancy between academic theory and practice then the theory must be modified.

It may sometimes happen that the designer comes across a peculiar structure or peculiar system of loading which is outside his previous experience. He is then forced back on analysis for lack of other information. In such cases an application of the simple elastic theory should be made and adjusted in accordance with the designer's experiences of other structures. For example, the author has put forward a modified theorem of three moments, a table of long columns, a theory of pressure distribution under raft foundations, and a revised table of moments in flat-slabs. These are based on actual structures. In all cases the structures were designed first and the analysis put forward afterwards.

Development of the Designer.—A French Engineer with extensive experience of design in both Europe and America once said to the author that he had noticed that all designers passed through three stages. In the first stage, when the designer had little knowledge and no confidence, he was reduced to copying other peoples designs and taking values of bending moments, etc., suggested by the older designers. After a little time he began to learn a little mathematics and began to understand the simple elastic theory. This marked the beginning of the second stage.

During this stage the designer becomes obsessed with the academic side. He plunges deeper into mathematical analysis ; works out all his results to four or five figures and adopts the most complicated types of statically indeterminate structures whenever possible. After a while he begins to notice that other designers can produce cheaper structures than he can. At first he is inclined to attribute this to their ignorance and neglect of important secondary stresses, but comes finally to realize that there is a wide gulf between the present theory of elastic structures and reinforced concrete. He then enters the third stage. In this stage the designer spends little time in computing moments or stresses but concentrates on producing the simplest type of structure, and on a sound arrangement of reinforcement, as he knows that these are the essential points. After roughing out a few figures he will go through the design and cut or increase sections which his experience has led him to believe are over strong or weak. The Engineer who gave vent to these sentiments thought that it was a pity, since the third stage so closely resembles the first, that it was not possible to omit the second stage altogether. The author cannot quite agree with this last remark. Familiarity with the elastic theory gives facility in dealing with figures and can do no harm so long as one realizes that it is merely an academic theory. Moreover a comparison between this theory and practice soon leads to a much closer understanding of the latter by contrast with the former. On the whole it is as well to let the budding designer exercise his mathematics (all the long analyses in this book were written out by the author during his first two years experience) as by doing so he will soon come to see where he makes mistakes.

Future Developments in Design.—If modern design were based on ultimate strength methods, many of the inconsistencies would vanish. A comparison between Tables RB I to RB VII and the "steel-beam" theory may be cited. Now as the stresses reach their ultimate value the intensity of compressive stress is constant near the top edge and the value of E may drop to 1,500,000. The stress in the compression steel is therefore about 20 times the value of f_c instead of only perhaps 8 or 9 times as given by the straight-line law. In short, the steel-beam theory for moderate percentages of steel gives a truer value of the ultimate bending strength than the straight-line theory. This also explains why columns under combined direct load and bending may be worked to a higher calculated stress (such calculation being based on the straight-line theory). The reactions between beams and columns also grow simpler as the ultimate strength is approached as the T-beams lose their continuity due to shear cracks and tend to split into separate spans. At ultimate load a T-beam of many spans is approximately a series of isolated spans supported on extended column-heads, which explains why it is impossible to adhere strictly to the theory of a continuous beam as propounded in Chapter X.

The most pressing need in modern design is some standardization of the method of calculating shear reinforcement. When a structure fails it is difficult to say which part failed first and which parts were broken in the fall. In some such cases it is easy to confuse cause and effect. Of all the failures of which the author has had first-hand knowledge he is convinced that shear failure was the cause. The principles laid down in Chapter IX represent the methods of design in modern use but are only safe when applied within limits. It is quite possible to design a beam in accordance with the principles in the first part of Chapter IX which will be weak in shear. In fact it appears that whereas beams in competitive design have usually a factor of safety of three against bending they have a factor of safety of two against shear. It is unfortunate that steel design should have been so fully exploited before reinforced concrete design developed.

In building work the designer has always had to compete with steel designs and in endeavouring to reduce the sizes of his beams to something approaching the size of the steel beams he has had to sacrifice some of the shear strength. As a result of this the weakest part of modern slab-and-girder floors is the shear strength of the T-beams, more particularly the main beams and most particularly discontinuous main beams. In the author's opinion the best possible method of reinforcing against shear is by bars bent up at a small angle running well over the supports as shown in Fig. 276 I. Even when the beam reaches its ultimate load and cracks badly, such a bar must continue to resist shear until it too reaches its ultimate tensile strength. This shape of bar was largely used in the Hennebique system. For a beam having heavier shear the arrangement shown in Fig. 276 II might be adopted. The idea in both cases is that the beam will still resist shear even after it has started to disintegrate so long as the bars are well anchored. It is usual when a haunch is provided to bend the bottom bars down into it. The author likes to run the main bars straight through and provide extra bars in the bracket. This means extra steel, which rather excludes the idea from competitive design. The arrangement

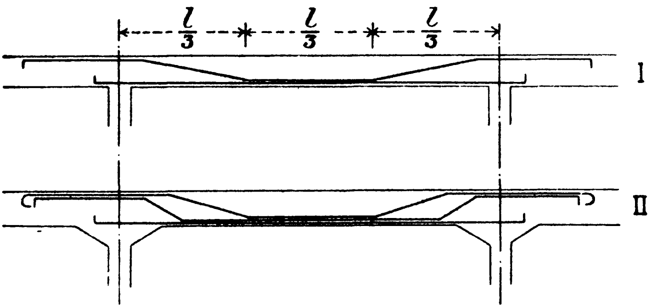

Fig. 276.

of shear bars shown in Fig. 276 I may be extended to long shallow beams where a bend at the third points gives too small an angle of slope, by imagining the beam to consist of a number of beams superimposed and supported on one another. Thus in Fig. 277 I we may imagine beam (*a*) is supported on beam (*b*), beam (*b*) on (*c*) and the whole when combined make beam (*d*). Any bent-up bars which are relied on for shear must be well anchored at the top

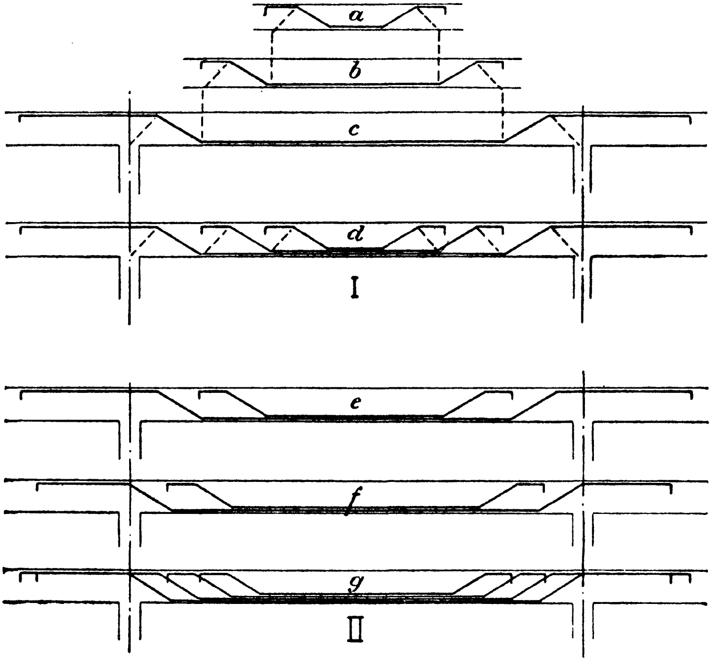

Fig. 277.

or their ultimate shear strength when the beam begins to crack, will be much reduced as they will pull out before developing their ultimate tensile strength. Extending this idea in Fig. 277 II we may consider that beam (*e*) and beam (*f*) when combined, give beam (*g*) which has a " continuous " system of bent-up bars the strength of which at any point is the sum of the strengths of the bar systems in beams (*e*) and (*f*). Comparing beam (*d*) which is a system of " single " bars, with beam (*g*) which is a " continuous " system, we see that the horizontal spacing in beam (*g*) is half that in beam (*d*). The strength of systems which have intermediate spacings is not clear. There is a tendency among some modern designers to bend bars up at steep angles. They do this in an attempt to supply tensile steel to cover all the areas shown in Table CB V, hoping thereby to increase the strength of the beam in bending. This is a mistake. Practical T-beams are amply strong in bending and comparatively weak in shear. The designer must concentrate on an arrangement of bars which gives the maximum shear strength. To increase the bending strength at the expense of the shear strength is to reduce the load-carrying capacity of the beam. The main function of stirrups is to tie the bent bars in and prevent local disintegration of the beam. No beam should be without them but it seems doubtful if a heavy system of stirrups will greatly increase the ultimate strength. Stirrups in beams are very much like links in columns. A certain percentage seems necessary to prevent brittleness or local failure but higher percentages, although increasing the resistance to sudden loads or shocks do not increase the ultimate strength under steadily applied loads. Some designers appear to think that " arching action " in beams gives additional strength. This arching action is merely the transference of compressive stress from the bottom flange near the support to the top flange at mid-span and is fully explained and covered by the ordinary straightforward theory of bending. In applying the results of Chapter IX, if the total shear on a beam approaches 200 $b' j d$ then about 120 $b' j d$ should be carried on bars inclined at a small angle and carried well past the supports. In arranging the stirrups the designer must think of the ultimate strength of the beam as a whole and not confine his attention to designing separate sections. Some kind of uniform continuous arrangement is wanted, or the main bars will pull the concrete to pieces. In contemplating the ultimate strength of a member it would seem that the stresses due to shrinkage and temperature can play little part once shear cracks have begun to appear, and this confirms what standard practice has shown for many years, i.e., that in most members these stresses may be neglected.

One possible solution of the problem presented by the weakness of main beams in shear is the adoption of beams whose section increases steadily from mid-span to support. Such beams could be designed for moments of say $\frac{w l^2}{24}$ at mid-span and $\frac{w l^2}{9}$ at the supports as the depth of construction would widen the support width into a large

" column head." Such beams would have greater shear strength for the same total volume of concrete.

The addition of stirrups to a beam having no bent-up bars undoubtedly increases the shear strength but the addition of stirrups to beams having bent-up bars does not seem quite so effective as the calculations in Chapter IX would show.

If the designer cannot find a suitable bar arrangement then the depth of the beam must be increased, as the provision of a very heavy stirrup system is a doubtful source of strength.

Cut Designs.—It may be necessary to reduce the cost of a design by reducing the quantities. This may arise through very keen competition to secure a low tender, or attempts to retrieve a very low and unprofitable tender which has been based on faulty pricing or a mistake in the original quantities. When this second reason is further complicated by the necessity to produce satisfactory calculations of strength to lay before the owners or governing authority, then desperate measures are called for. The first step is to examine the superloads, earth pressures, etc. allowed for. These should be reduced to a reasonable minimum. The designer should then try, by examining the spacing and arrangement of members, to arrive at a more economical type of structure. The next step is to increase the allowable stresses for bending without reducing the shear strength. If these methods fail, then it is by far the best policy to leave the matter where it is and either lose the chance of securing the contract or put up with the loss on a contract already secured, and make good resolutions for the future. Structures which collapse are expensive in themselves and provide unsavoury advertisements.

In cutting the quantities on a contract already secured remember that floor slabs are stronger than calculated and collapse of floor slabs is practically unknown. In cutting down steel quantities reduce the numbers or diameters of bars but do not unduly reduce their lengths. Long and continuous bars are the essence of ultimate strength. In submitting calculations to surveyors, etc., who represent some authority, remember that these officials are generally men of experience, and a submission of all the facts of the case will generally meet with consideration. Most officials are willing to take a sensible rather than a literal interpretation of the various regulations and bye-laws.

Future Development of the Designer.—Concrete, despite all our efforts, is still the most variable of all building materials. A " four-inch " floor slab having a theoretical *d* of 3·3125 ins. may easily with reasonably good workmanship, be only 3¾ ins. thick with a *d* of 2·5 in. (an error of 25%). It is on official record that an " 8½-in. " reinforced concrete floor varied between 7·5 ins. and 9·8 ins. (Bulletin No. 106, Univ. Illinois.)

The man who attempts to estimate stresses to four or five figures is deceiving himself and wasting his employers' time. The errors that do occur in drawing offices (particularly in properly run and properly organized offices) are confined to errors in policy and errors of omission or forgetfulness. The most common cases are due to choice of a wrong steel factor and omission of one or more complete items from a bill of quantities due to an oversight. The first of these may be reduced by experience or by drawing out the bars themselves in doubtful cases. The second may be reduced by more careful preparation of preliminary drawings. The author knows of one case where six different people (including the author) examined the preliminary drawing and quantities for a small job. One important item was omitted from the quantities but not one of the six noticed it. The members in question, although essential to the structure, were not shown on any view contained in the original drawing. Had this drawing contained an elevation of the structure the omission would have been impossible. Vital mistakes in calculations, except in the case of rare and peculiar structures, are not to be feared as the designer's common sense will soon tell him if his sizes are far wrong.

The one essential virtue is to obtain a quick and comprehensive grasp of the whole problem. In practical design speed is accuracy. It is the man who pays long and laborious attention to detail who makes most serious mistakes.

The beginner must adopt methods of design which are in general vogue. These methods are somewhat inconsistent but are not to be changed except by a slow process of evolution. Above all things the beginner must pin his faith to standard practice and actual structures. All theories and text-books should be examined with the utmost scepticism. Written matter which does not coincide with the facts as embodied in work actually carried out, should be treated as mere conjecture unworthy of consideration. The drawings and details published in the technical press should be carefully and critically examined particularly those produced in competitive design. Any curves or tables required to expedite design should be drawn up by the person using them or at least checked through. Such curves must be simple and expressed in as few lines as possible. The basis on which they have been calculated must be indicated clearly on each one. Mathematical analysis should be worked out in spare time and reduced to such a form that it can be applied immediately to a practical case. The author has found mathematical calculation quicker and more reliable than graphics in most cases. The mathematics involved is straightforward integration of standard forms which can be turned up immediately in any good text-book such as Lamb's *Infinitesimal Calculus*. It is not so difficult to arrive at results as it is to express them in easy terms and arrange them to be easily applied. Aim at giving the final result directly. If a curve is to show the bending strength of a member don't drag in intermediate terms showing the neutral axis or lever arm if these can possibly be omitted. Before commencing a design all drawings and documents connected therewith should be read through and the process should be repeated after the design is finished. Speed is the keynote of good and accurate design. An intelligent guess is far superior to a long and bungled calculation.

Principle of Greatest Strength.—This is based on the fact that the elastic moduli decrease as the ultimate load is approached. If one section is overstressed it weakens and thereby transfers load to other portions whenever possible. Most practical engineers will endorse the theory in certain small applications. The statement is as follows:

" Take any reinforced concrete structure and introduce any arbitrary number of frictionless hinges (points of contra-flexure). If the structure can be demonstrated to be safe after the operation, then the original structure is also safe."

If not, then let it be otherwise. Let the structure with hinges be safe and the structure without hinges be unsafe. Take the structure with hinges and increase the bending strength at all hinges until the original structure is reproduced. By increasing the bending strength at various points we have rendered the structure unsafe, which is contrary to common sense.

This theory concerns the ultimate strength. Application of it indiscriminately may lead to extensive cracking at working stresses. It is not suitable therefore for hydraulic structures.

As the theory presumes some plastic yielding of the reinforcement it should not be applied to structures built with non-ductile steel of any kind.

Future Developments.—Designing for Permanence.—Many of the troubles of the reinforced concrete industry are due to the fact that it is to a large extent divided into separate parts out of touch with one another. We have academic research workers who know nothing of drawing-office practice nor of construction ; designers who are never allowed to visit contracts and who also know nothing of construction ; contractors' men who know nothing of the difficulties of the designer nor the relative importance of the various processes of making concrete and fixing steel. Consequently many designs and specifications (and unfortunately some Regulations) pay no regard to the difficulties of placing concrete, making shuttering or fixing steel. In general this results in insufficient cover, insufficient clearances between bars, arrangements of steel that are not self-supporting and expensive shuttering. These lead to honeycombing, air pockets between the bars, rusting and blowing of the steel, and general deterioration when exposed to the weather. Of all cases of deterioration which the author has seen, more than half were due, in the first instance, to faults in design. Unless all designers are prepared to study the principles of construction (see the author's *Construction in Reinforced Concrete*) no improvement in this respect can be hoped for.

Another point that needs earnest attention is the shrinkage of modern cements. We must either have cements with less effective shrinkage or we must pay more attention to shrinkage stresses in design. At present designers pay great attention to calculating structural stresses, and, as a last thought, add a little steel at random to cover shrinkage and temperature stresses. As the shrinkage stresses are obviously often much higher than stresses due to loading it would be more logical, in many cases, to make some attempt to calculate the shrinkage stresses and, as a last thought, add a little steel at random to cover structural stresses.

Welding.—In some cases during the last thirty years where the reinforcement was particularly heavy or the spans particularly long, welding was adopted, not always with the happiest results. In very heavy construction, welding, by eliminating overlaps, hooks and wasted grip lengths allows the use of larger bars more closely spaced thus greatly increasing the maximum possible percentage of reinforcement. Modern methods of electric welding are a great improvement on the old but the best system the author has yet seen still depends entirely on the personal skill of the operator to produce a reasonably efficient weld. The bars should be sawn at the ends as the distortion caused by commercial shearing tends to spoil the welds. A rigidly connected cage of reinforcement is not an unmixed blessing and it seems that there is little likelihood of a general adoption of welding on small and medium-sized contracts in the near future. Some engineers will accept a weld whose strength is equal to the yield point stress in the bar but the author does not like this outlook. It is advisable for both the designer and contractor to have some knowledge of practical welding before deciding on its use. Adoption of welding will not change the principles of design. Apart from the question of cost there would be no appreciable structural advantage in using welding on the vast majority of contracts.

Vibration—For many years pre-cast units have been made on vibrated tables and attempts have been made to introduce vibration for *in situ* work but the author has seen no method which was equal to really good hand tamping. Unless thoroughly understood, carefully applied and continuously supervised, vibration may prove calamitous. If an efficient and foolproof method of vibrating concrete into position could be devised we could reduce the clearances between bars and work to higher stresses on the concrete but the main basis of design would remain unchanged. Vibration has been applied very successfully to making of pre-cast piles.

APPENDIX A.*

CHICAGO REINFORCED CONCRETE FLAT-SLAB RULING AMENDED

1. **Definitions.**—Flat slabs as understood by this ruling are reinforced-concrete slabs, supported directly on reinforced columns with or without plates or capitals at the top, the whole construction being hingeless and monolithic without any visible beams or girders. The construction may be such as to admit the use of hollow panels in the ceiling or smooth ceiling with depressed panels in the floor.

2. The column capital shall be defined as the gradual flaring out of the top of the column without any marked offset.

3. The drop panel shall be defined as a square or rectangular depression around the column capital extending below the slab adjacent to it.

4. The panel length shall be defined as the distance c. to c. of columns of the side of a square panel, or the average distance c. to c. of columns of the long and short sides of a rectangular panel.

5. **Columns.**—The least dimension of any concrete column shall be not less than one-twelfth the panel length, nor one-twelfth the clear height of the column.

6. **Slab Thickness.**—The minimum total thickness of the slab in inches shall be determined by the formula : $t = W^{\frac{1}{2}}/44$ (= square root of W divided by 44), where t = total thickness of slab in inches, W = total live load and dead load in pounds on the panel, measured c. to c. of columns.

7. In no case shall the thickness be less than $\frac{1}{32}$ of the panel length ($L/32$) for floors, nor $\frac{1}{40}$ of the panel length ($L/40$) for roofs (L being the distance c. to c. columns).

8. In no case shall the thickness of slab be less than 6 ins. for floors or roofs.

9. **Column Capital.**—When used the diameter of the column capital shall be measured where its vertical thickness is at least $1\frac{1}{2}$ ins. and shall be at least 0·225 of the panel length.

The slope of the column capital shall nowhere make an angle with the vertical of more than 45 deg. Special attention shall be given to the design of the column capital in considering eccentric loads, and the effect of wind upon the structure.

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* Hool and Johnson *Concrete Engineers' Handbook*.

10. **Drop Panel.**—When used, the drop panel shall be square or circular for square panels and rectangular or elliptical for oblong panels.

11. The length of the drop shall not be less than one-third of the panel length ($L/3$) if square, and not less than one-third of the long or short side of the panel respectively, if rectangular.

12. The depth of the drop panel shall be determined by computing it as a beam, using the negative moment over the column capital specified elsewhere in this ruling.

13. In no case, however, shall the dimensions of the drop panel be less than required for punching shear along its perimeter, using the allowable unit shearing stresses specified below.

14. **Shearing Stresses.**—The allowable unit punching shear on the perimeter of the column capital shall be $\frac{3}{50}$ of the ultimate compressive strength of the concrete as given in Sect. 533 of the building ordinance. The allowable unit shear on the perimeter of the drop panel shall be 0·03 of the ultimate compressive strength of the concrete. In computing shearing stress for the purpose of determining the resistance to diagonal tension the method specified by the ordinance shall be used.

15. **Panel Strips.**—For the purpose of establishing the bending moments and the resisting moments of a square panel, the panel shall be divided into strips known as strip *A* and strip *B*. Strip *A* shall include the reinforcement and slab in a width extending from the centre line of the columns for a distance each side of this centre line equal to one-quarter of the panel length. Strip *B* shall include the reinforcement and slab in the half width remaining in the centre of the panel. At right angles to these strips, the panel shall be divided into similar strips *A* and *B*, having the same widths and relations to the centre line of the columns as the above strips. These strips shall be for designing purposes only, and are not intended as the boundary lines of any bands of steel used.

16. These strips shall apply to the system of reinforcement in which the reinforcing bars are placed parallel and at right angles to the centre line of the columns, hereinafter known as the two-way system, and also to the system of reinforcement in which the reinforcing bars are placed parallel, at right angles to and diagonal to the centre line of the columns hereinafter known as the four-way system.

17. Any other system of reinforcement in which the reinforcing bars are placed in circular, concentric rings and radial bars, or systems with steel rods arranged in any manner whatsoever, shall comply with the requirements of either the two-way or the four-way system herein specified.

18. **Bending Moment Co-efficients, Interior Panel, Two-way System.**—In panels where standard drops and column capitals are used as above specified, the negative bending moment, taken at a

cross-section of each strip *A* at the edge of the column capital or over it, shall be taken as $WL/30$.

19. The positive bending moment taken at a cross-section of each strip *A* midway between column centres shall be taken as $WL/60$.

20. The positive bending moment taken at a cross-section of each strip *B* in the middle of the panel shall be taken as $WL/120$.

21. The negative bending moment taken at a cross-section of each strip *B* on the centre line of the columns shall be taken as $WL/120$.

22. In the formulas hereinabove given W = total live and dead load on the whole panel in pounds, L = panel length, c. to c. of columns.

23. **Bending Moment Coefficients, Interior Panels, Four-way System.**—In panels where standard drops and column capitals are used as above specified, the negative bending moment, taken at a cross-section of each strip *A* at the edge of column capital or over it, shall be taken as $WL/30$.

24. The positive bending moment, taken at a cross-section of each strip *A*, midway between column centres, shall be taken as $WL/80$.

25. The positive bending moment, taken at a cross-section of each strip *B*, taken in the middle of the panel, shall be taken as $WL/120$.

26. The negative bending moment, taken at a cross-section of each strip *B* on the centre line of the columns, shall be taken as $WL/120$.

27. **Bending Moment Coefficients, Wall Panels.**—Where wall panels with standard drops and capitals are carried by columns and girders built in walls, as in skeleton construction, the same coefficients shall be used as for an interior panel, except as follows: The positive bending moments on strips *A* and *B* midway between wall and first line of columns shall be increased 25%.

28. Where wall panels are carried on new brick walls, these shall be laid in Portland cement mortar and shall be stiffened with pilasters as follows: If a 16-in. wall is used, it shall have a 4-in. pilaster. If a 12-in. wall is used, it shall have an 8-in. pilaster. The length of pilasters shall be not less than the diameter of the column, nor less than one-eighth of the distance between pilasters. The pilasters shall be located opposite the columns as nearly as practicable, and shall be corbelled out 4 ins. at the top, starting at the level of the base of the column capital. Not less than 8-in. bearing shall be provided for the slab, the full length of wall.

The coefficients of bending moments required for these panels shall be the same as those for the interior panels except as provided herewith: The positive bending moments on strips *A* and *B* midway between the wall and first line of columns shall be increased 50%.

29. Where wall panels are supported on old brick walls, there shall be columns with standard drops and capitals built against the wall, which shall be tied to the same in an approved manner, and at least an 8-in. bearing provided for the slab, the full length. Where this is impracticable, there shall be built a beam on the underside of slab adjacent to the wall between columns, strong enough to carry 25% of the panel load.

The coefficients of bending moments for the two cases of slab support herein described shall be the same as those specified in Sect. 27 and Sect. 28 for skeleton and wall bearing condition, respectively.

30. Nothing specified above shall be construed as applying to a case of slabs merely resting on walls or ledges, without any condition of restraint. These shall be figured as in ordinary beam-and-girder construction specified in the ordinances.

31. **Bending Moment Coefficients, Wall and Interior Columns.**—Wall columns in skeleton construction shall be designed to resist a bending moment of $WL/60$ at floors and $WL/30$ at roof. The amount of steel required for this moment shall be independent of that required to carry the direct load. It shall be placed as near the surfaces of the column as practicable on the tension sides, and the rods shall be continuous in crossing from one side to another. The length of rods below the base of the capital and above the floor line shall be sufficient to develop their strength through bond, but not less than 40 diameters, nor less than one-third the clear height between the floor line and the base of the column capital.

32. The interior columns must be analysed for the worst condition of unbalanced loading. It is the intention of this ruling to cover ordinary cases of eccentric loads on the columns by the requirement of Sect. 5. Where the minimum size of column therein specified is found insufficient, however, the effect of the resulting bending moment shall be properly divided between the adjoining slab and the columns above and below according to best principles of engineering, and the columns enlarged sufficiently to carry the load safely.

33. **Bending Moment Coefficients, Panels Without Drops, or Capitals, or Both.**—In square panels where no column capital or no depressions are used, the sum total of positive and negative bending moments shall be equal to that computed by the following formula

$$B.M. = (WL/8)\,(1{\cdot}53 - 4\,k + 4{\cdot}18\,k^3)$$

where $B.M.$ = Numerical sum of positive and negative bending moments regardless of algebraic signs ;

W = Total live and dead load on the whole panel ;

L = Length of side of a square panel, c. to c. of columns ;

k = Ratio of the radius of the column or column capital to panel length, L.

This total bending moment shall be divided between the positive and the negative moments in the same proportion as in the typical square panels for two-way or four-way systems specified above for interior and wall panels respectively.

34. **Point of Inflection.**—For the purpose of making the calculations of the bending moment at the sections away from the column capital, the point of inflection shall be considered as being one-quarter the distance c. to c. of columns, both crosswise and diagonally, from the centre of the column.

35. **Tensile Stress in Steel and Compressive Stress in Concrete.**—The tensile stress in steel and the compressive stress in the concrete to resist the bending moment shall be calculated on the basis of the reinforcement and slab in the width included in a given strip, and according to the assumptions and requirements given in Sects. 532 to 535 inclusive of the building ordinance. The steel shall be considered as being concentrated at the centre of gravity of all the bands of steel in a given strip.

36. For the four-way system of reinforcement the amount of steel to resist the negative bending moment over the support in each strip *A* shall be taken as the sum of the areas of steel in one cross band and one diagonal band. The amount of steel to resist the positive bending moment of each strip *B* shall be considered as the area of the steel in a diagonal band. The amount of steel to resist the positive bending moment in each strip *A* shall be considered as the area of the steel in a cross band, and the amount of steel to resist the negative moment in each strip *B* shall be the steel included in the width of strip *B*.

37. For the two-way system of reinforcement the amount of steel to resist the bending moment in any strip shall be considered as the area of steel included in the width of the strip.

38. In both systems of reinforcement the compressive stress in the concrete in any strip shall be calculated by taking the area of steel considered for each strip and applying it in a beam formula based on the principles of Sect. 535 of the building ordinance.

39. Where drop panels are used, the width of beam assumed to resist the compressive stresses over the column capital shall be the width of the drop.

40. The width of beam, where no drop panels are used, shall be the width of steel bands. Where this is found insufficient, the area shall be increased by introducing compression steel in the bottom of slab.

41. **Rectangular Panels.**—When the length of panel in either two-way or four-way system does not exceed the breadth by more than 5%, all computations shall be based on a square panel whose side equals the mean of the length and breadth, and the steel equally distributed among the strips according to the coefficients above specified.

42. In no rectangular panel shall the length exceed the breadth by more than one-third the latter.

43. **Rectangular Panels, Four-way System.**—In the four-way system of reinforcement, where length exceeds breadth by more than 5%, the amount of steel required in strip *A*, long direction, both positive and negative, shall be the same as that required for the same strip in a square panel whose length is equal to the long side of the rectangular panel.

44. The amount of steel, strip *A*, short direction, positive and negative, shall be the same as that required for the same strip in a square panel, whose length is equal to the short side of the rectangular panel.

45. The amount of steel in strip *B*, positive and negative, shall be the same as that required for similar strip in a square panel whose length is equal to the mean of the long and the short side of the rectangular panel.

46. In no case shall the amount of steel in the short side be less than two-thirds of that required for the long side.

47. **Rectangular Panels, Two-way System.**—In the two-way system of reinforcement the amount of steel required for the positive and the negative moment of each strip *A* shall be determined in the same manner as indicated for the four-way system above.

48. The amount of steel in strip *B*, positive and negative, running in short direction, shall be equal to that required for the same strip in a square panel whose length equals the long side of the rectangular panel.

49. The amount of steel in strip *B*, long direction, positive and negative, shall be equal to that required for the same strip in a square panel, whose length equals the short side of the rectangular panel.

50. In no case shall the amount of steel in strip *B*, long direction, be less than two-thirds of that in the short direction.

51. **Walls and Openings.**—Girders and beams shall be constructed under walls, around openings and to carry concentrated loads.

52. **Spandrel Beams.**—The spandrel beams or girders shall, in addition to their own weight and the weight of the spandrel wall, be assumed to carry 20% of the wall panel load uniformly distributed upon them.

53. **Placing of Steel.**—In order that the slab bars shall be maintained in the position shown in the design during the work of pouring the slab, spacers and supports shall be provided satisfactory to the Commissioner of Buildings. All bars shall be secured in place at intersections by wire or other metal fastenings. In no case shall the spacing of the bars exceed 9 ins. The steel to resist the negative moment in each strip *B* shall extend one-quarter of the panel length beyond the centre line of the columns in both directions.

54. Splices in bars may be made wherever convenient, but preferably at points of minimum stress. The length of splice beyond the centre point, in each direction, shall not be less than 40 diameters of the bars, nor less than 2 feet. The splicing of adjacent bars shall be avoided as far as possible.

55. Slab bars which are lapped over the column, the sectional area of both being included in the calculations for negative moment, shall extend not less than 0·25 of the panel length for cross bands and 0·35 of the panel length for diagonal bands, beyond the column centre.

56. **Computations.**—Complete computations of interior and wall panels and such other portions of the building as may be required by the Commissioner of Buildings shall be left in the office of the Commissioner of Buildings when plans are presented for approval.

57. **Test of Workmanship.**—The Commissioner of Buildings or his representative may choose any two adjacent panels in the building for the purpose of ascertaining the character of workmanship. The test shall not be made sooner than the time required for the cement to set thoroughly, nor less than 6 weeks after the concrete had been poured.

58. All deflections under test load shall be taken at the centre of the slab, and shall be measured from the normal unloaded position of the slab. The two panels selected shall be uniformly loaded over their entire area with a load equal to the dead load plus twice the live load, thus obtaining twice the total design load. The load shall remain in place not less than 24 hours. If the total deflection in the centre of the panel under the test load does not exceed $\frac{1}{800}$ of the panel length, the slab may be placarded to carry the full design live load. If it exceeds this amount of deflection, and recovers not less than 80% of the total deflection within 7 days after the load is removed, the slab may be placarded to carry the full design live load. If the deflection exceeds the allowable amount above specified, and the recovery is less than 80% in 7 days after the removal of the test load, other tests shall be made on the same or other panels, the results of which will determine the amount of live load the slabs will be permitted to carry.

59. **General.**—The design and the execution of the work shall conform to the general provisions and the spirit of the Chicago Building Ordinances in points not covered by this Ruling and to the best engineering practice in general.

60. **Enforcement.**—This Ruling shall go into effect on and after January 1, 1918. All previous rulings on flat slabs are hereby rescinded.

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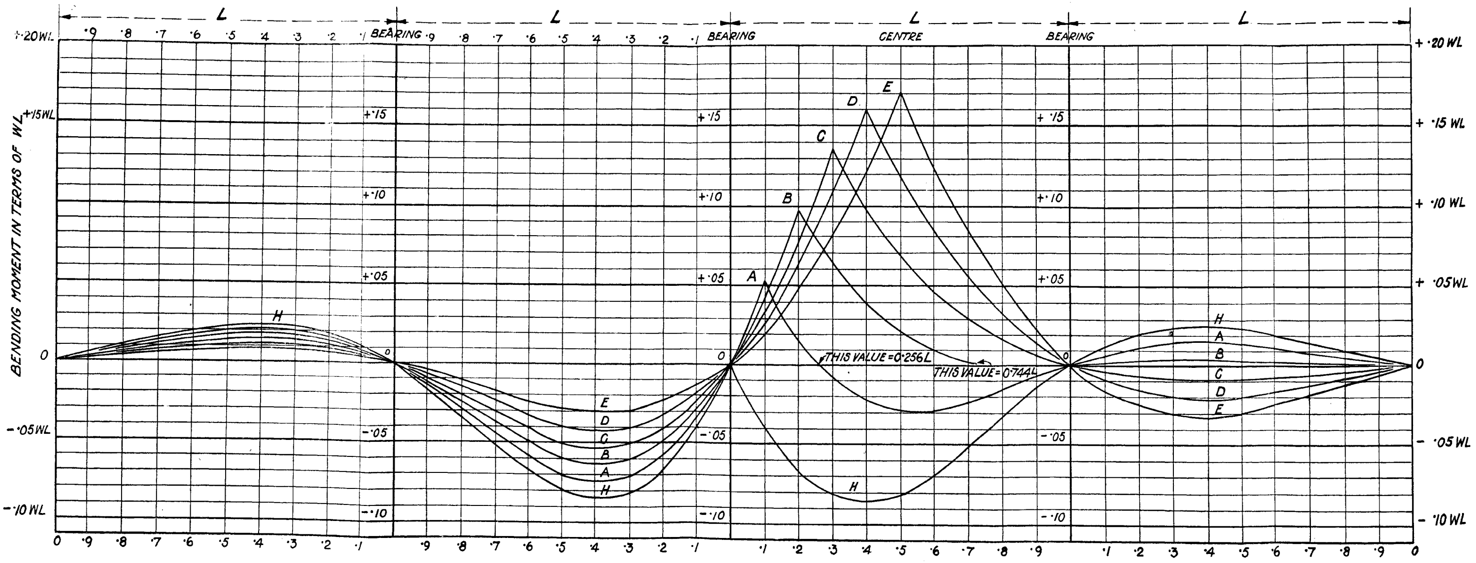
PLATE I
CENTRE SPANS

IDEAL ELASTIC CONTINUOUS BEAM. INFINITE NUMBER OF EQUAL SPANS (EI=CONSTANT), FREE SUPPORTS



BENDING MOMENT INFLUENCE LINES FOR MOVING POINT LOAD W (see Chapter X).

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